Optimal Monetary Policy in Small Open Economies: Producer Currency Pricing

Mikhail Dmitriev† and Jonathan Hoddenbagh‡

First Draft: November 2012 This Version: October 2019

We establish the share of exports in production as a sufficient statistic for optimal non-cooperative monetary policy. Under financial autarky, markups positively co-move with the export share. For complete markets, markups should be procyclical if the export share is procyclical. When central banks cooperate, markups are constant under complete markets, and countercyclical under financial autarky.

**Keywords**: Open economy macroeconomics; Optimal monetary policy; Price stability.

**JEL Classification Numbers**: E50, F41, F42.

---

*This is a substantially revised and updated edition of the October 2014 and June 2013 version of the paper. We thank Fabio Ghironi, Susanto Basu, Pierpaolo Benigno, Giancarlo Corsetti, Eyal Dvir, Peter Ireland and two anonymous referees for very helpful comments, as well as seminar participants at Boston College, the BC/BU Green Line Macro Meeting, the Eastern Economic Association Meeting and the Canadian Economic Association Meeting. Any errors are our own.

†Florida State University. E-mail: mdmitriev@fsu.edu.
‡Johns Hopkins University. E-mail: jon.hoddenbagh@gmail.com.
1 Introduction

Although price stability is widely viewed as a benchmark monetary policy for central banks, various ingredients in the open economy drive optimal policy away from replicating the flexible price allocation. Cooperation in the absence of commitment (Rogoff, 1985), imperfect risk-sharing (Corsetti, Dedola, and Leduc, 2010), incomplete exchange rate pass-through (Devereux and Engel, 2003), non-cooperative policy (De Paoli, 2009a,b), and trade elasticities (Benigno and Benigno, 2003) generate deviations from price stability. Because there are so many additional ingredients in the open economy relative to the closed economy, it is very difficult to suggest a one-size fits all optimal monetary policy like price stability. The precise optimal policy is sensitive to a variety of assumptions and specific parameter settings.

Our goal is to provide a coherent, tractable framework to examine optimal monetary policy in small open economies under producer currency pricing. We establish a set of simple rules to guide central banks in four unique cases where monetary policy is either cooperative or non-cooperative and markets are complete or cross-border trade in financial assets is prohibited. These four cases nest most of the key distortions in the open economy: nominal rigidities, terms of trade externalities and incomplete cross-country risk-sharing.

For optimal non-cooperative policy, we find that central banks should generate markups that follow the share of exported goods in total production, unless these markup movements cause excess consumption volatility. For example, if the share of exported goods is procyclical, then monetary policy should generate procyclical markups. For optimal cooperative policy, markups should be constant when markets are complete, and countercyclical under financial autarky.

The paper makes several contributions to the literature. First, we provide a unified framework for studying cooperative and non-cooperative optimal monetary policy under both complete markets and financial autarky. Second, our solution is analytical and covers the full space of parameters instead of focusing on a particular calibration. Third, we are the first to study cooperative policy for small open economies under producer currency pricing (PCP) and show how it differs from the non-cooperative case. Fourth, in our study of the optimal policy, we do not restrict import and export trade elasticities to be equal to each other. Finally, we establish the export share of GDP as a sufficient statistic for non-cooperative policy.

We consider a continuum of small open economies that are hit by asymmetric productivity shocks,
following Gali and Monacelli (2005). We deviate from their paper in three ways. First, we do not restrict our analysis to the widely used Cole-Obstfeld (1991) specification where the coefficient of relative risk aversion and trade elasticities are set to unitary values but instead analyze the most general case analytically. Second, we extend their analysis to both cooperative and non-cooperative policies under financial autarky. Finally, we utilize one period in advance price rigidities used by Obstfeld and Rogoff (2000, 2002), Corsetti and Pesenti (2001, 2005), Faia and Monacelli (2008) Egorov and Mukhin (2019), Dmitriev and Hoddenbagh (2019), instead of a more traditional Calvo setup.

We think about monetary policy in terms of deviating from the flexible price allocation using markups. For example, when the policymaker intends to decrease markups, she lowers the interest rate and depreciates the currency. Producer prices remain stable in the home currency and fall when expressed in foreign currencies. Thus, export volume increases. At the same time, import prices remain constant in foreign currencies and rise in the home currency. Therefore, the terms of trade depreciate. Also, locally produced goods become more competitive at home and crowd out imports in terms of volumes. Consumer prices, composed of higher import prices and stable local product prices, increase. Finally, output and employment, driven by higher demand for exports and domestic import substitution, tend to go up, raising wages and reducing markups. Thus, negative deviations of markups from the steady state are associated with expansionary monetary policy, which generates positive output gaps, currency depreciation, and price and wage inflation.

We begin our analysis by considering cooperative policy and complete markets. In this case, nominal rigidity is the only distortion present and optimal monetary policy replicates the flexible price allocation through a policy of price stability. While we are the first to consider cooperative policy under complete markets for small open economies, our contribution for this specification is mostly technical. For example, Benigno and Benigno (2006), and Corsetti, Dedola, and Leduc (2010) have established that replicating the flexible price allocation is optimal for cooperative policymakers under complete markets for two large open economies.

We then consider optimal cooperative policy under financial autarky. In this case, we have only two distortions: nominal rigidities and incomplete risk-sharing across countries. We show that for empirically relevant parameter settings, monetary policy generates countercyclical markups. These markup movements are designed to manipulate the terms of trade and redistribute resources from countries with positive supply shocks to countries hit by adverse shocks through terms of trade de-
preciation for the former and appreciation for the latter. As trade elasticities rise and monopoly power at the export level deteriorates, central banks lose their ability to influence the terms of trade, such that they focus more on replicating flexible prices and less on terms of trade adjustments.

The closest relevant study by Corsetti, Dedola, Leduc (2010) is for large open economies. They set import and export elasticities to be equal. While their focus is when optimal policy replicates the flexible price allocation or the first best allocation, we study the full markup dynamics.

Next, we analyze optimal policy under financial autarky without cooperation. There are three distortions that drive the equilibrium away from the efficient allocation: nominal rigidities, terms of trade externalities, and market incompleteness. We give an explicit analytic expression for the optimal markup, and then establish the share of exports in production as a sufficient statistic for the optimal monetary policy. Optimal markups positively comove with the export share. If the export share is constant, replication of the flexible price allocation is optimal. Indeed, once the policy becomes non-cooperative, risk-sharing across countries becomes irrelevant for the policymaker. She desires to sell exported goods at a positive markup and have no markup for the products produced and consumed at home. Moreover, when prices are set one period in advance, policymakers cannot influence markups systematically. As a result, central banks tend to increase (decrease) markups when the export share goes up (down).

De Paoli (2009a) also analyzes non-cooperative policy under financial autarky for the limiting case of two large open economies. We differ from her analysis in several ways. First, her study has a more quantitative focus so that she fixes most of the parameters to particular values and uses Calvo pricing. Instead, we consider markup movements for the broadest range of parameters. Second, we provide a sufficient statistic for optimal monetary policy: the policymaker only needs the dynamics of the export share regardless of the underlying parameters. Third, we allow export and import elasticities to differ from each other.

Finally, we consider the case of complete markets and non-cooperative policy. In this setting the optimal markup is procyclical whenever the export share is procyclical. A procyclical markup enables the policymaker to extract higher monopolistic rents from foreigners through terms of trade appreciation which stabilizes domestic consumption. On the other hand, when the export share is countercyclical, countercyclical markups generate stronger terms of trade externality rents and destabilize consumption. When the costs from destabilizing consumption exceed the benefits from the terms of trade externality, the optimal markups might be procyclical despite the countercyclical export share.
Non-cooperative policy under complete markets and PCP has been studied by De Paoli (2009a, 2009b) and Faia and Monacelli (2008) for small open economies as the limiting case of two large economies. As before, we differ from these studies by differentiating between trade elasticities and by deriving analytical expressions for markups and other variables, instead of focusing on a particular calibration with Calvo pricing. We also consider a case where trade elasticities equal to each other.

Using the export share as a sufficient statistic helps to explain why under the Cole-Obstfeld (1991) specification, where trade elasticities and risk-aversion are set to one, the flexible price allocation is optimal under all four cases considered here. Under Cole-Obstfeld the export share is constant and terms of trade movement provide complete risk-sharing. As a result, policymakers have no incentive to stabilize consumption or extract monopolistic rents from foreigners, as the export share is constant in all cases. Also, our principle explains why under non-cooperative policy the flexible price allocation is optimal for fully open economies as the export share remains equal to one over the cycle. The export share becomes less relevant for non-cooperative policy only if the central bank loses its ability to influence the terms of trade. Then the optimal policy is to replicate the flexible price allocation.

There are several reasons why we use producer currency pricing, despite some recent empirical evidence supporting dominant currency pricing or DCP (Goldberg and Tille, 2008; Gopinath et al., 2010; Gopinath et al., 2016). First, both PCP and DCP imply equal sensitivity of import prices to the exchange rate. Second, the evidence by Amity, Itskhoki, and Konings (2014) on export prices supports at least fifty percent exchange rate pass-through for large import-intensive exporters and full pass-through for smaller exporters, while DCP assumes no pass-through for export prices. Third, under PCP high export elasticities allow terms of trade to be stable and independent from monetary policy, similar to DCP. Finally, PCP, formally developed by Mundell (1963), Fleming (1962), Obstfeld and Rogoff (1995), and others, serves as the benchmark for optimal monetary policy analysis since other setups add extra distortions in addition to the ones present in PCP.

We differ from many papers in the field by having prices set one period in advance instead of Calvo pricing. This price setting allows us to arrive at the optimal conditions in a fully non-linear manner. We linearize the equilibrium afterward, which makes our results robust to the Kim and Kim (2003) critique of linear-quadratic approximation. Also, the Calvo approach to nominal rigidities introduces price dispersion in addition to standard output gap costs. Although price dispersion increases the complexity of the model, it is proportional to the output gap, which makes economic intuition for optimal

\[\text{Benigno and Woodford (2012) establish conditions under which linear-quadratic analysis is correct and robust to the Kim and Kim (2003) criticism.}\]
policy similar between Calvo pricing and one period in advance pricing. As our contribution is more analytical than quantitative, we use one period in advance price-setting for the sake of tractability.

We also differ from the optimal policy literature by studying a continuum of small open economies instead of considering a limit of two economies. Having the continuum allows us to differentiate import and export elasticities, which are equal in the case of two countries. In the absence of this differentiation, the analysis faces empirical and theoretical difficulties. For example, under inelastic export demand, an increase in the quantity of exports leads to lower export revenues, which makes the positive supply shocks potentially welfare-reversing. Second, under inelastic export demand with constant elasticity, an infinite export tax generates limitless export revenues and consumption. To avoid these problems, we set the export elasticity to be above one, and allow the import elasticity to vary between zero and infinity. Forcing import and export elasticities to be equal allows us to nest the limiting case of two open economies. Also, a continuum of small open economies under cooperation nests two large open economies as a special case. For instance, if we divide the countries in a continuum into two groups and allow them to have two realizations of the technology shocks, the framework is equivalent to two large open economies as small open economies differ only by the realization of the technology shock. While we do not introduce symmetric shocks, the division of the asymmetric shocks into two groups of positive and negative values allows for a close approximation of two large open economies.

Our research is also closely related to the old debate on the value of trade elasticities in the data. While the trade literature finds larger elasticities, with values ranging between four and five (Anderson and Van Wincoop, 2004; Simonovska and Waugh, 2014a,b; Imbs and Mejean, 2015), the international macro and finance literature (a non-exhaustive list starting from Backus et al., 1994; Stockman and Tesar, 1995; and many others) often assumes these elasticities are smaller, with values ranging between 0.8 and 1.5. This debate has a strong effect on optimal monetary policy where trade elasticities play a central role. We show that this debate is not relevant for the policymaker since the effect of trade elasticities on monetary policy is expressed through the dynamics of the directly observed export share.

---

2 Infinite export tariffs are never optimal for two large open economies, where they generate a negative wealth effect through impoverishing of the trading partner and provoke trade wars. However, potential welfare reversals under inelastic export demand are still present even for two large open economies.

3 Non-cooperative policy for two large open economies differs from the non-cooperative policy under a continuum due to the presence of strategic interactions.
2 The Model

We consider a continuum of small open economies represented by the unit interval, as popularized in the literature by Gali and Monacelli (2005, 2008). Each economy consists of a representative household and a representative firm. All countries are identical ex-ante: they have the same preferences, technology, and price-setting. Ex-post, economies differ depending on the realization of their technology shock. Households are immobile across countries, while goods can move freely across borders. Each economy produces one final good, over which it exercises a degree of monopoly power. In particular, countries are able to manipulate their terms of trade even though they are measure zero, similar to an individual producer in a model of monopolistic competition. However, because countries are small, they have no impact on world income or the world interest rate.

We use a one-period-in-advance price setting to introduce nominal rigidities. Monopolistic firms set the next period’s nominal prices in terms of the domestic currency, before the next period’s production and consumption decisions. These firms charge a constant markup in the flexible price equilibrium, utilizing their monopoly power at the firm level. Given this preset price, firms supply as much output as demanded by households.

We lay out a general framework below and then focus on four particular cases: cooperative policy under complete markets and financial autarky, and non-cooperative policy under complete markets and financial autarky.

Households  In each economy \( i \in [0, 1] \), there is a representative household with lifetime expected utility

\[
E_t \left\{ \sum_{k=0}^{\infty} \beta^k \left( \frac{c_{it+k}^{1-\sigma}}{1-\sigma} - \frac{N_{it+k}^{1+\varphi}}{1+\varphi} \right) \right\},
\]

where \( \beta < 1 \) is the household discount factor, \( C \) is the consumption basket, and \( N \) is household labor effort. Households face a general budget constraint that nests both complete markets and financial autarky; we will discuss the differences between the two in subsequent sections. For now, it is sufficient to simply write out the most general form of the budget constraint:

\[
C_{it} = (1 + \tau_i) \left( \frac{W_{it}}{P_{it}} \right) N_{it} + \frac{P_{F,i} \cdot \Pi_{it}}{P_{it}} \cdot D_{it} + \Pi_{it} - \tau_{it} + \frac{B_{it}}{P_{it}} - \frac{B_{it+1}}{R_{it} P_{it}}.
\]
The distortionary subsidy on household labor income in country $i$ is denoted by $\tau_i$, while $T_{it}$ is a lump-sum tax collected from the households to finance this subsidy. Overall, the government budget is balanced at every period. These subsidies and taxes are designed to enforce the efficient steady state allocation. $\Pi_{it}$ denotes profits from the monopolistic firms which is distributed lump-sum to households. Without loss of generality, we assume that equities in the model are not traded. The consumer price index corresponds to $P_{it}$, while the nominal wage is $W_{it}$. The price index $P_{F,it}$ reflects the price of the basket composed from the imported goods expressed in units of currency $i$. $D_{it}$ denotes net state-contingent portfolio payments expressed in the units of the imported goods, which is available to households under complete markets. This portfolio consists of state-contingent bonds, available for every state of the world. To simplify the exposition, we allow only domestic households to hold non-contingent bonds $B_{it}$. This is not a limitation as foreigners have access to the portfolio of state-contingent securities. In equilibrium, non-contingent bonds are relevant only as a source of information to pin down interest rate dynamics. When international asset markets are complete, households perform all cross-border trades in contingent claims in period 0, insuring against all possible states in all future periods. Under financial autarky, households have access only to non-contingent bonds so that $D_{it} = 0$.

**Consumption and Price Indices** The consumption basket for a representative small open economy $i$, consists of home goods $C_{H,it}$, and foreign goods, denoted by $C_{F,it}$. It is defined as follows:

$$C_{it} = \left[ (1 - \alpha)^{\frac{1}{\eta}} (C_{H,it})^{\frac{\eta+1}{\eta}} + \alpha^{\frac{1}{\eta}} (C_{F,it})^{\frac{\eta+1}{\eta}} \right]^\frac{\eta}{\eta+1},$$

where the import basket $C_{F,it}$ is defined as

$$C_{F,it} = \left( \int_0^1 (C_{F,ijt})^{\frac{\eta+1}{\eta}} dj \right)^\frac{\eta}{\eta+1}.$$  

The consumer price index $P_{it}$ is an aggregator of the domestic variety price $P_{H,it}$ and import price $P_{F,it}$.
index $P_{F,it}$:

$$
P_{it} = \left[ (1 - \alpha)P_{H,it}^{1-\eta} + \alpha P_{F,it}^{1-\eta} \right]^{\frac{1}{1-\gamma}}. 
$$

(5)

Here the import price index $P_{F,it}$ is a constant elasticity of substitution aggregator that takes the following form:

$$
P_{F,it} = \left( \int_0^1 (\mathcal{E}_{ijt} P_{H,jt})^{1-\gamma} dj \right)^{\frac{1}{1-\gamma}},
$$

(6)

where the variety produced in country $j$ is sold in country $i$ for a price $\mathcal{E}_{ijt} P_{H,jt}$. We define the nominal bilateral exchange rate $\mathcal{E}_{ijt}$ as units of currency $i$ per one unit of currency $j$.

Household expenditure minimization yields the demand for the home variety $C_{H,it}$, demand for imported varieties $C_{F,it}$, and the relative demand for the variety produced in country $j$ and consumed in country $i$:

$$
C_{H,it} = (1 - \alpha) \left( \frac{P_{H,it}}{P_{it}} \right)^{-\eta} C_{it},
$$

(7)

$$
C_{F,it} = \alpha \left( \frac{P_{F,it}}{P_{it}} \right)^{-\eta} C_{it},
$$

(8)

$$
C_{F,ijt} = \left( \frac{P_{H,jt}}{P_{F,jt}} \right)^{-\gamma} C_{F,it}.
$$

(9)

We assume that producer currency pricing (PCP) holds so that the law of one price (LOP) applies. Put differently, the price of the same good is equal across countries when converted into a common currency. In this case, the good produced in country $i$ has a price in country $j$ equal to $\mathcal{E}_{ijt} P_{H,it}$. Although the law of one price holds for individual varieties, purchasing power parity does not hold because of home bias in consumption. Nevertheless, import price indices are identical across countries when converted to the same currency, such that $P_{F,it} = \mathcal{E}_{ijt} P_{F,jt}$. The terms of trade for country $j$, defined as the home currency price of exports over the home currency price of imports, is denoted $\tilde{P}_{H,jt} = \frac{P_{H,jt}}{P_{F,jt}}$. We define the aggregate consumer price index normalized by the import price index as $\tilde{P}_{it} = \frac{P_{it}}{P_{F,it}}$. In our model, log deviations of $\tilde{P}_{it}$ from the steady state correspond to real exchange rate movements to a first order approximation. Using the normalized price levels $\tilde{P}_{H,it}$ and $\tilde{P}_{it}$, we modify
the demand expressions (10), (11), and (12) into

\[ C_{H,it} = (1 - \alpha) \tilde{P}_{H,it}^{-\eta} \tilde{P}_{it}^\eta C_{it}, \]  
\[ C_{F,it} = \alpha C_{it} \tilde{P}_{it}^\eta, \]  
\[ C_{F,ijt} = \tilde{P}_{H,it}^{-\gamma} C_{F,it}. \]  

Goods market clearing requires that the supply of the domestic variety produced in country \(i\) equals demand from home consumers \(C_{H,it}\) and foreign consumers \(\int_0^1 C_{F,jit} \, dj\):

\[ Y_{it} = C_{H,it} + \int_0^1 C_{F,jit} \, dj. \]  

Substituting equations (10) and (12) into the goods market clearing condition (13) yields global demand for country \(i\)'s unique variety:

\[ Y_{it} = (1 - \alpha) \left( \frac{P_{H,it}}{P_{it}} \right)^{-\eta} C_{it} + \left( \frac{P_{H,it}}{P_{F,it}} \right)^{-\gamma} \int_0^1 C_{F,jit} \, dj. \]  

Given the symmetric structure of the model as well as the independence of idiosyncratic shocks across countries, the integral on the import basket in (14) is equivalent to the unconditional expectation, which corresponds to the ergodic mean of the import basket. More formally:

\[ \int_0^1 C_{F,jit} \, dj = \mathbb{E}\{C_{F,it}\} = \alpha \mathbb{E}\left\{ \left( \frac{P_{F,it}}{P_{it}} \right)^{-\eta} C_{it} \right\} = \alpha \mathbb{E}\{P_{it}^\eta C_{it}\}. \]  

Non-cooperative policymakers take global import \(\int_0^1 C_{F,jit} \, dj\) as given. Substituting (15) into (14) yields the following goods market clearing condition:

\[ Y_{it} = (1 - \alpha) \tilde{P}_{H,it}^{-\eta} \tilde{P}_{it}^\eta C_{it} + \alpha \tilde{P}_{H,it}^{-\gamma} \mathbb{E}\{\tilde{P}_{it}^\eta C_{it}\}. \]  

We define the share of the domestically produced variety in economy \(i\) that is exported goods as:

\[ E_{s,it} = \frac{\alpha \tilde{P}_{H,it}^{-\gamma} \mathbb{E}\{\tilde{P}_{it}^\eta C_{it}\}}{Y_{it}}. \]  

**Production** Each economy \(i\) consists of a group of intermediate goods producers, \(h \in [0, 1]\), who exercise monopoly power over their unique variety, and a perfectly competitive final goods producer,
who aggregates the intermediates in a constant elasticity of substitution fashion into a final good. For simplicity, we assume that intermediates are non-tradable. Thus, each country bundles its intermediates into one final good, which is consumed both at home and abroad.\(^5\)

Production of intermediates requires technology \(Z_{it}\), which is common across firms within a country, and labor \(N_{it}(h)\), which is unique to each firm. We assume that technology is independent across time and across countries (assumptions that can be easily relaxed), but is identical across firms within the same country. Given this, the production function of a representative intermediate goods firm \(h\) in country \(i\) is \(y_{it}(h) = Z_{it}N_{it}(h)\), and aggregate output is described by

\[
Y_{it} = Z_{it}N_{it}. \tag{18}
\]

Because intermediate goods firms produce differentiated varieties, they exercise monopoly power and charge markups over their costs. A perfectly competitive final goods producer aggregates the intermediate input of each firm in the following way:

\[
Y_{it} = \left[ \int_0^1 Y_{it}(h)^{\frac{\epsilon+1}{\epsilon}} \, dh \right]^{\frac{1}{\epsilon+1}}, \tag{19}
\]

where \(\epsilon\) is the elasticity of substitution between different intermediates. For country \(i\), the price of the final good, \(P_{H, it}\), is a function of the nominal price for intermediate goods, \(P_{H, it}(h): P_{H, it} = \left[ \int_0^1 P_{H, it}(h)^{1-\epsilon} \, dh \right]^{\frac{1}{1-\epsilon}}\). Cost minimization by the perfectly competitive final goods exporter leads to the following demand for intermediate variety \(h\):

\[
Y_{it}(h) = \left[ \frac{P_{H, it}(h)}{P_{H, it}} \right]^{-\epsilon} Y_{it}. \tag{20}
\]

Intermediate goods firms choose the profit maximizing price for their unique good one-period-in-

---

\(^5\)We assume non-tradable intermediates with a final tradable consumption good that aggregates those intermediates for simplicity. In Gali and Monacelli’s (2005, 2008) setup, intermediate goods are tradable, such that every country’s import consumption basket is made up of an infinite number of varieties imported from an infinite number of countries. This assumption requires integrating over two continuums. While it is straightforward for us to maintain their setup, we prefer the tractable alternative: a final goods producer bundles the domestically produced intermediates for export. In this way, each country produces only one unique variety, and we only need to integrate over one continuum. This assumption does not change the results in any way. In both cases the household consumption basket in each country is made up of imported goods from all \(i\) countries, which are themselves made up of intermediates produced domestically.
advance according to the following condition:

\[ P_{H,it}(h) = \mu \frac{E_{t-1} \left\{ C_{it}^{-\sigma} Y_{it}(h) \frac{W_{it}}{Z_{it} P_{it}} \right\}}{E_{t-1} \left\{ C_{it}^{\sigma} Y_{it}(h) \frac{P_{it}}{P_{it}} \right\}}, \]  

(21)

where \( \mu = \frac{\varepsilon}{\varepsilon - 1} \) is the markup, which defines the degree of monopoly power at the firm level.

Households maximize utility (1) subject to their budget constraint (2). The first order condition with respect to labor gives the following household labor supply:

\[ \frac{W_{it}}{P_{it}} = \frac{1}{1 + \tau} N_{it}^{\varphi} C_{it}^{\sigma}. \]  

(22)

Because firms are identical at the national level, in equilibrium \( P_{H,it}(h) = P_{H,it} \) and \( Y_{it}(h) = Z_{it} N_{it} = Y_{it} \). To eliminate steady state markups at the firm level and achieve the first best steady state, we choose distortionary labor subsidies such that \( \frac{\mu}{1 + \tau} = 1 \). Using the optimal pricing equation (21), the labor supply condition (22), and the fact that prices are preset at time \( t - 1 \), the optimal pricing condition under PCP is:

\[ 1 = \frac{E_{t-1} \left\{ N_{it}^{1+\varphi} \right\}}{E_{t-1} \left\{ C_{it}^{-\sigma} N_{it} Z_{it} P_{H,it} P_{it} \right\}}. \]  

(23)

We evaluate monetary policy in relation to markup fluctuations. In our setup, flexible prices are equivalent to constant markups. As marginal costs for one unit of the final good are equal to \( \frac{W_{it}}{Z_{it}} \), we define the markup as:

\[ \mu_{it} = \frac{P_{H,it}}{W_{it}}. \]  

(24)

Substituting the labor supply condition (22), the terms of trade \( \tilde{P}_{H,it} = \frac{P_{H,it}}{P_{F,it}} \) into (24), we obtain:

\[ \mu_{it} = \frac{Z_{it} \tilde{P}_{H,it}}{N_{it}^{\varphi} C_{it}^{\sigma}}. \]  

(25)

Note, that under flexible prices we set \( \mu_{it} = 1 \), which fully corresponds to the optimal pricing condition (23) once we take out the expectations. The intuitive interpretation of (23) is that firms set up prices equal to expected marginal costs. Optimal pricing condition can also be formulated in terms
of markups. Plugging (25) into (23) gives:

\[ 1 = \frac{E_{t-1}\{N_{it}^{1+\varphi}\}}{E_{t-1}\{N_{it}^{1+\varphi} \mu_{it}\}}. \]  

(26)

Intuitively, equation (26) says that markups on average are equal to one. In other words, the central bank chooses cyclical markup dynamics around one to maximize household welfare subject to market clearing constraints.

**Complete Markets**

In complete markets, agents in each economy have access to a full set of domestic and foreign state-contingent assets to insure against country-specific consumption risk. Households in all countries maximize their lifetime utility (1) choosing consumption, labor, and a complete set of nominal state-contingent portfolio payments, subject to the budget constraint (2). Since countries are symmetric ex-ante, complete markets imply the following risk-sharing condition

\[ \frac{C_{it}^{-\sigma}}{\bar{P}_{it}} = \frac{C_{jt}^{-\sigma}}{\bar{P}_{jt}} \quad \forall i, j \]  

(27)

which states that the marginal utility from consumption of imported varieties \( C_{F,it} \), which is equal to the ratio of marginal utility of consumption and normalized aggregate price index, must be equal across all countries.

When international asset markets are complete, households perform all cross-border trades in contingent claims in period zero before the realization of any shocks, insuring against all possible states in all future periods. To ensure that there are no Ponzi schemes in issuing state-contingent securities, we impose the intertemporal asset constraint that all transactions in period zero before the realization of shocks must be balanced. Payment for claims issued in subsequent periods must equal payment for claims received. In Appendix A, we show that the intertemporal asset constraint for complete markets is:

\[ \mathbb{E}\left\{ \sum_{t=1}^{\infty} \beta^t C_{it}^{-\sigma} \frac{D_{it}}{\bar{P}_{it}} \right\} = 0. \]  

(28)

which corresponds to equation (A.6). Intuitively, the intertemporal asset constraint stipulates that the present discounted value of future earnings should be equal to the present discounted value of future consumption flows. Total state-contingent portfolio payments across countries in every period must
sum to zero such that in the absence of world-wide uncertainty:

$$E\{\bar{P}_{it}C_{it}\} = E\{\bar{P}_{H,it}Y_{it}\}. \quad (29)$$

In Appendix A we combine the risk-sharing condition (27), and balanced portfolio flows among all countries in the absence of symmetric shocks (29) to yield the following expression for consumption in country $i$:

$$E \left\{ \bar{P}_{it}^{\sigma - 1} \bar{P}_{it}^{\frac{1}{2}} C_{it} \right\} = E \{ \bar{P}_{H,it} Y_{it} \}, \quad (30)$$

which corresponds to (A.9).

Our treatment of complete markets appears somewhat different from the rest of the literature. For example, Gali and Monacelli (2005) do not use the intertemporal asset constraint (28). As a result, they assume that the value $C_{it}^{\sigma - 1} \bar{P}_{it}$ is constant and independent from monetary policy, which is not correct. While the expression $C_{it}^{\sigma - 1} \bar{P}_{it}$ is indeed independent of the realization of the shocks, our derivations show that it is a composite of ergodic means of the endogenous variables, which are impacted not only by the monetary policy but can potentially have a first-order impact on the optimal policy rule itself.

**Financial Autarky**

In financial autarky there is no trade in state-contingent financial assets, such that $D_{it} = 0 \ \forall i, t$. The aggregate resource constraint under financial autarky specifies that the nominal value of output in the home country must equal the nominal value of consumption in the home country:

$$P_{it}C_{it} = P_{H,it}Y_{it}. \quad (31)$$

Normalizing this expression by $P_{F,it}$ gives:

$$\bar{P}_{it}C_{it} = \bar{P}_{H,it}Y_{it}. \quad (32)$$
**Technology Shocks**

We assume that technology shocks are independent and identical across time and countries, and have a log-normal distribution such that

\[
\log Z_{it} \sim \mathcal{N}(0, \sigma^2_z).
\]  

(33)

The model is formulated such that independence across time can be relaxed. It is also straightforward to relax the assumption of independence across countries by introducing a global aggregate component to technology. Our conclusions are robust to assumption of \textit{iid} exogenous shock dynamics.

### 3 Optimal Monetary Policy

**Setting Up the Optimization Problem**

Without loss of generality, we assume a cashless limiting economy. Central banks use an interest rate rule to set monetary policy, which affects the dynamics of the nominal exchange rate through uncovered interest rate parity. Under PCP, fluctuations in the nominal exchange rate pass through fully to import prices expressed in domestic currency \( P_{F,it} \). As the price of the final domestic good is fixed one period in advance, nominal exchange rate fluctuations affect the terms of trade \( \bar{P}_{H,it} \). The terms of trade, in turn, affect the real exchange rate, consumption, and hours worked. Ultimately, a change in hours worked affects household disutility from labor, wages, and markups. We refer the reader to Appendix D, where the relationship between the interest rate and the terms of trade is formally established.

The timing of the model is described in Figure 1 below. Before any shocks are realized, national central banks declare their policy for all states of the world. With this knowledge in hand, households lay out a state-contingent plan for consumption, labor hours, money, and asset holdings. After that, shocks hit the economy. Note that under financial autarky, no international asset trading occurs.
We summarize the optimization problem for the four cases we consider below. In each economy, the central bank maximizes the utility of the representative household

$$\max_{\tilde{P}_{H,t}, P_t, C_t, N_t} \mathbb{E}_t \left\{ \sum_{t=1}^{\infty} \beta^t \left[ \frac{C^{1-\sigma}_t - N_t^{1+\phi}}{1 - \sigma} \right] \right\},$$

subject to the optimal pricing condition (35), goods market clearing (36), aggregate consumer price index (37), and asset market clearing (38). The first order conditions with respect to the terms of trade $\tilde{P}_{H,t}$, the normalized price index $\tilde{P}_t$, consumption $C_t$, and labor $N_t$ are:

$$E_{t-1} \left\{ C_t^{-\sigma} N_t Z_t \tilde{P}_{H,t} / \tilde{P}_t \right\} = E_{t-1} \left\{ N_t^{1+\phi} \right\},$$

$$Z_t N_t = (1 - \alpha) \tilde{P}^{-1-\eta}_t C_t \tilde{P}^{\eta}_{FF} + \alpha \tilde{P}^{-\gamma}_{H,t} (1 - \mathbb{1}_{CP}) \mathbb{E}[C_t \tilde{P}^{\eta}_t] + (1 - \mathbb{1}_{CP}) C_F,$$

$$\tilde{P}^{1-\eta}_t = (1 - \alpha) \tilde{P}^{1-\eta}_{H,t} + \alpha,$$

$$C_t = \mathbb{1}_{CM} \mathbb{E} \left\{ \frac{Z_t N_t \tilde{P}_{H,t}}{\tilde{P}_{t}^{1-\sigma}} \right\} + (1 - \mathbb{1}_{CM}) \frac{Z_t N_t \tilde{P}_{H,t}}{\tilde{P}_t}.$$

Without loss of generality subscript $i$ is omitted in expressions (34)-(38) as the economies effectively differ only by the realization of the technology shock. The indicator function $\mathbb{1}_{CP}$ is equal to one when the policy is cooperative, and zero otherwise. The indicator function $\mathbb{1}_{CM}$ is equal to one when financial markets are complete, and zero under financial autarky. When the policy is cooperative, the domestic central bank takes into account the effect its policy on other countries through the average consumption of foreign goods $\int_0^1 C_{F,i,t} \, di = EC_{F,i,t} = E[C_t \tilde{P}_t]$. On the other hand, when the policy is non-cooperative, central banks take aggregate world consumption of foreign goods as given so that

$$\int_0^1 C_{F,i,t} \, di = C_F.$$

Stochastic processes for the terms of trade $\tilde{P}_{H,t}$ and the technology shock $Z_t$ fully define the equilibrium path for the endogenous variables $\tilde{P}_t, C_t, N_t$ using (35)-(38). Under the optimal monetary policy, the terms of trade $\tilde{P}_{H,t}$ reacts endogenously to the technology shock $Z_t$ so that all variables
\( \tilde{P}_{H,t}, \tilde{P}_t, C_t, N_t \) can be expressed as functions of the technology shock. Before we analyze optimal monetary policy, let us consider a few stochastic processes for the terms of trade.

**Stochastic Processes for the Terms of Trade**

Traditionally monetary policy in closed economies is expressed using nominal interest rate rules such as the Taylor rule. In our model, the interest rate affects the rest of the economy through the terms of trade. For analytical convenience, we consider shocks to the terms of trade instead of shocks to the interest rate rule. In particular, we employ a class of interest rate paths that generate the following stochastic process for the terms of trade:

\[
\tilde{P}_{H,t} = f(Z_t). \tag{39}
\]

We focus on interest rate rules that are not history dependent, where the terms of trade react only to the technology shock in the current period. The log-linear equivalent of the process described by (39) is:

\[
\hat{\tilde{P}}_{H,t} = \alpha_T Z_t. \tag{40}
\]

After we combine (39) with the constraints (35)-(38), we can express the dynamics of the log-linearized endogenous variables (\( \hat{Y}_t, \hat{C}_t, \hat{N}_t, \tilde{P}_t, \hat{\mu}_t \)) in terms of the technology shock. The search for the optimal monetary policy rule is equivalent to searching for the coefficient \( \alpha_T \) that gives the highest welfare. To compare different policy rules and values for \( \alpha_T \), we need to know how terms of trade fluctuations impact other endogenous variables for a given technology shock. Lemma 1 below summarizes the impact of the terms of terms of trade on other key endogenous variables for a given technology shock.

**Lemma 1** For an equilibrium that is characterized by (35)-(39) and for a given realization of the technology shock, a monetary policy that generates lower (higher) terms of trade also leads to lower (higher) markups, and higher (lower) consumption, employment, and output. This result holds under financial autarky and complete markets, for cooperative and non-cooperative policies.

**Proof** See Appendix C.1.

Lemma 1 describes the effect of the terms of trade depreciation on the economy for a given technology shock. The transmission mechanism from expansionary monetary policy through terms of trade.
trade depreciation into markups, consumption, hours worked and output is intuitive. Lower terms of trade lead to an increase in consumption, employment, and output through higher exports and import substitution, as currency depreciation makes domestic goods more competitive at home and abroad. Higher demand for labor drives up wages, and in the presence of price rigidities reduces markups.

It is natural to consider the stochastic process for the terms of trade that can replicate the flexible price allocation.

**Lemma 2** Under the flexible price allocation, a positive technology shock increases consumption and output and leads to a decline in the terms of trade and a real exchange rate depreciation. This result holds under both financial autarky and complete markets.

**Proof** See Appendix C.2. ■

Lemma 2 states that under flexible prices, a positive technology shock increases consumption and output, similar to a closed economy. An increase in productivity also leads to a decrease in the terms of trade as abundant home produced goods become cheaper relative to foreign goods.

The optimal monetary policy under some circumstances differs from replication of the flexible price allocation. We use markup dynamics to characterize monetary policy, while in the literature monetary policy is often characterized by the importance of flexible prices or exchange rate stabilization. In the corollary that follows, we show that the policymaker, who desires to increase markups relative to the flexible price allocation, should appreciate the currency, increase terms of trade, and decrease the output gap.

**Lemma 3** Under both financial autarky and complete markets, a monetary policy that generates procyclical (countercyclical) markups also gives rise to a countercyclical (procyclical) output gap and a more (less) stable real exchange rate.

**Proof** Lemma 1 establishes a monotonic relationship between the terms of trade, markups, and output. Since the output gap is zero when markups are equal to one, and for any given state of the world higher markups imply lower output, markups above one necessarily imply that output gaps are negative, and the terms of trade and real exchange rate are lower than they would be under flexible prices. ■

To summarize, Lemma 1, 2, and 3 state that the model we consider does not generate any non-standard dynamics. The impact of monetary policy or productivity shocks are consistent with standard
closed or open economy models. With these mechanisms established for simple monetary rules, we are now ready to consider optimal monetary policy.

Complete Markets and Cooperative Policy

In this section, we examine the optimal monetary policy for cooperative central banks under complete markets. In this setup, nominal rigidity is the only distortion present. Therefore, replicating flexible prices is an optimal policy that also implements the efficient allocation, which is stated more formally in the proposition below.

Proposition 1  In complete markets, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where \( \mathbb{1}_{CP} = 1 \) and \( \mathbb{1}_{CM} = 1 \). The indicators for cooperative policy \( \mathbb{1}_{CP} \) and complete markets \( \mathbb{1}_{CM} \) are set to one. The solution is:

\[
\hat{\mu}_t = 0
\]

The resulting equilibrium allocation exactly coincides with the flexible price allocation. Mimicking the flexible price allocation is the optimal policy under cooperation, and corresponds to the social planner allocation.

Proof  See Appendix B.1. ■

In our setup, constant markups imply flexible prices. Moreover, since labor subsidies remove monopolistic distortions, and as complete markets provide full risk-sharing, the resulting allocation is efficient. The finding that the flexible price allocation is the optimal policy under cooperation, and corresponds to the social planner allocation, aligns with Benigno and Benigno (2006), and Corsetti, Dedola, and Leduc’s (2010) results for two large economies. Our contribution here is mostly technical.

Financial Autarky and Cooperative Policy

In this section, we consider the optimal cooperative monetary policy under financial autarky. To our knowledge, we are the first to consider this case for small open economies. Corsetti, Dedola, and Leduc (2010) consider cooperative policy under financial autarky for two open economies. Before comparing our results, we formulate the optimal cooperative policy problem under financial autarky.

Proposition 2  In financial autarky, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where \( \mathbb{1}_{CP} = 1 \) and \( \mathbb{1}_{CM} = 0 \). The indicators for cooperative policy \( \mathbb{1}_{CP} \) and complete
markets $1_{CM}$ are set to one and zero, respectively. The solution is given by

\[ \hat{\mu}_t = \frac{G_2}{F_2} \zeta_t, \quad (41) \]

where

\[ G_2 = -\alpha(1 + \varphi)((1 - \alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)). \]

\[ F_2 = \varphi + \sigma + \alpha\eta + \alpha\gamma - 2\eta\sigma + \eta^2\sigma + \gamma^2\sigma + \alpha^2\eta^2\sigma - 2\gamma\sigma - \alpha^2\eta - 2\alpha\eta\sigma + 2\eta\gamma\sigma + 2\alpha\eta \]

\[ - 2\alpha\varphi - 2\gamma\varphi + \alpha^2\varphi + \eta^2\varphi + \gamma^2\varphi + \alpha^2\eta^2\varphi + 4\alpha\eta\varphi + 2\alpha\gamma\varphi + 2\eta\gamma\varphi - 2\alpha\eta^2\varphi - 2\alpha^2\eta\varphi - 2\alpha\eta\gamma\varphi. \]

The resulting equilibrium allocation differs from the flexible price allocation.

**Proof** See Appendix B.2. ■

The resulting markup movement reported in equation (41) is a complicated function of openness, trade elasticities, labor disutility, and risk-aversion. Although the monetary authority will deviate from replicating the flexible price allocation in general, there are four specific calibrations where the flexible price allocation will be optimal for the policymaker. Corollary 2.1 details each of these four cases.

**Corollary 2.1** Cooperative central banks under financial autarky implement the flexible price allocation, whenever any of the four listed conditions below is satisfied:

- **a)** Under Cole-Obstfeld conditions, when $\sigma = \gamma = \eta = 1$.
- **b)** Under full home bias, $\alpha = 0$.
- **c)** Whenever $(\eta\sigma - 1)(1 - \alpha) + \sigma(\gamma - 1) = 0$.
- **d)** For an economy with stable terms of trade, when $\gamma \to \infty$ or $\eta \to \infty$.

**Proof** Under a), b), and c) in equation (41) we have $G_2 = 0$, which implies constant markups and flexible prices. With respect to d), we know that as the export elasticity $\gamma$ or the import elasticity $\eta$ increase, the numerator $F_2$ grows faster than $G_2$. Or, more formally, the following relationships hold as export elasticity increases:

\[ \lim_{\gamma \to \infty} \frac{G_2}{\gamma} \to -\alpha(1 + \varphi)\sigma, \quad \lim_{\gamma \to \infty} \frac{F_2}{\gamma^2} \to \sigma + \varphi, \]  

thus \( \lim_{\gamma \to \infty} \frac{G_2}{F_2} = \hat{\mu}_t \to 0 \). Also, the following relationships apply as import elasticity grows:

\[ \lim_{\eta \to \infty} \frac{G_2}{\eta} \to -\alpha(1 - \alpha)(1 + \varphi)\sigma, \quad \lim_{\eta \to \infty} \frac{F_2}{\eta^2} \to (1 - \alpha)^2(\sigma + \varphi), \]  

thus \( \lim_{\eta \to \infty} \frac{G_2}{F_2} = \hat{\mu}_t \to 0 \). ■

Under cooperative policy and financial autarky policymakers face two distortions: nominal rigidities and incomplete cross-country risk-sharing. Optimal monetary policy in this setting thus faces
a tradeoff between mitigating the distortionary impact of nominal rigidities through price stability as well as the mitigation of incomplete cross-country risk-sharing via terms of trade adjustments. Under Cole-Obstfeld conditions, where $\sigma = \gamma = \eta = 1$, terms of trade movements provide full risk-sharing, such that the optimal monetary policy is to mimic the flexible price allocation and thereby eliminate distortions from nominal rigidities. In a closed economy ($\alpha = 0$), risk-sharing cannot be improved by trade flows, and the central bank thus focuses on eliminating the internal distortion arising from nominal rigidities via replication of the flexible price allocation. Finally, as the trade elasticities increase ($\eta, \gamma$), monopoly power at the national level is reduced, and policymakers are less able to influence the terms of trade through monetary policy. To summarize, policymakers focus on price stability when terms of trade movements provide full risk-sharing under flexible prices, or monetary policy is powerless to reduce this risk-sharing.

How do cooperative central banks improve risk-sharing across countries? Under cooperative monetary policy, countries with positive productivity shocks reduce their markups and depreciate their currencies and terms of trade via lower interest rates in order to supply exports for the rest of the world at reduced prices. On the other hand, countries with negative productivity shocks increase their interest rate, markups and terms of trade in order to sell exports at higher prices. This cooperative monetary policy response to asymmetric shocks stabilizes employment and consumption across countries.

Corollary 2.2 below summarizes formally the conditions required for countercyclical markups.

**Corollary 2.2** *If $\gamma > 1 - \eta(1 - \alpha) + \frac{1 - \alpha}{\sigma}$, then markup $\hat{\mu}_t$ negatively comoves with output.*

**Proof**

$$\hat{\mu}_t = -\alpha \frac{(1 - \alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)}{(1 - \alpha)^2(\eta - 1)^2 + (\gamma - 1)^2 + 2\eta\gamma(1 - \alpha) + 2\alpha\gamma - 1} \hat{Y}_t.$$ (42)

Since $\gamma \geq 1$, $\eta > 0$, and $\alpha > 0$, the denominator in equation (42) is positive. Indeed, the denominator monotonically increases with $\gamma$. One can also show that for $\gamma = 1$, the denominator is equal to $((1 - \alpha)\eta - \alpha)^2 \geq 0$. Therefore, the denominator is positive for $\gamma > 1$. Under $\gamma > 1 - \eta(1 - \alpha) + \frac{1 - \alpha}{\sigma}$, the numerator in the fraction is positive as well. Therefore, overall coefficient on the right hand side is negative. ■

For example, if $\sigma \geq 2$, then $\gamma \geq 1.5$ guarantees that, regardless of other parameters, the policymaker reduces markups in response to higher output, further boosting production. Since in the data
the trade elasticity $\eta$ is positive and risk-aversion $\sigma$ is $\geq 2$, markups will comove negatively with output for almost all parameter values. As a result, cooperative monetary policy in financial autarky yields countercyclical markups, procyclical output gaps, and manipulates the terms of trade to increase cross-border risk-sharing.

Corsetti, Dedola, and Leduc (2010) study cooperative policy under financial autarky for two open economies. We differ from their research in several dimensions. First, we use prices set one period in advance instead of Calvo pricing. Second, we allow trade elasticities to be different from each other. Third, they analyze first order conditions without solving fully for each allocation. For example, they find out that the optimal policy under financial autarky faces a trade-off between stabilizing output gaps, price dispersion, and global demand imbalances. However, their work does not provide an explicit roadmap for when the central bank should seek to depreciate or appreciate the exchange rate given certain parameter values.

Our analytical solution allows us to see the tradeoff between the output gap stabilization and imperfect risk-sharing explicitly. In particular, manipulation of the terms of trade should deliver risk-sharing, and the policymakers try to move international prices to deliver higher risk-sharing relative to maximizing output gaps. We solve for the explicit policy rule under central bank cooperation: countries with a positive shock lower their interest rate and depreciate their currency and terms of trade relative to the flexible price allocation in order to provide cheaper products to the rest of the world and stabilize consumption abroad.

Our analytical solution allows us to investigate the role of specific parameters. When trade elasticities are low ($\eta$ and $\gamma$ close to one), goods are less substitutable across countries, and central banks exert a stronger influence on the terms of trade because of monopoly power at the export level. However, under low trade elasticities the terms of trade also provide a high degree of risk-sharing across countries, mitigating the need for central banks to deviate from the flexible price allocation. On the other hand, as trade elasticities increase, the terms of trade provide less risk-sharing across countries without policy intervention, and thus policymakers will manipulate the terms of trade to improve risk-sharing and move the economy closer to the efficient allocation. The caveat is that as goods become more substitutable, policymaker’s capacity to influence the terms of trade declines. As a result, under high trade elasticities, central banks will focus more on closing national output gaps and replicating the flexible price allocation than improving risk-sharing. Overall, optimal monetary policy will be closer to the flexible price allocation for low and high trade elasticities, while deviations from
flexible prices will be strongest for intermediate values of the trade elasticities.

**Financial Autarky and Non-Cooperative Policy. Introduction of the Export Share.**

In this section, we study non-cooperative monetary policy under financial autarky. There are three distortions that may drive the equilibrium away from the first-best allocation. In addition to nominal rigidities and incomplete risk-sharing across countries, policymakers exploit terms of trade externalities to boost national welfare. Moreover, while incomplete cross-country risk-sharing moves the equilibrium away from the first best, this distortion only has an indirect effect on the policymaker, who is disinterested in providing cross-country risk-sharing. Nevertheless, market incompleteness affects the dynamics of the endogenous variables and has an effect on the trade-off the policymaker faces between terms of trade externalities and nominal rigidities. In general, the policymaker chooses to deviate from the flexible price equilibrium in this environment. Proposition 3 below formally establishes that optimal markups deviate from one, such that the flexible price allocation is suboptimal.

**Proposition 3** In financial autarky, non-cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where the indicators for cooperative policy \( \mathbb{1}_{CP} \) and complete markets \( \mathbb{1}_{CM} \) are set to zero. The solution is

\[
\hat{\mu}_t = \frac{G_3}{F_3} Z_t,
\]

where

\[
G_3 = -\alpha (1 - \alpha) (1 + \varphi) (\eta - 1) (\eta - 1 + \gamma),
\]
\[
F_3 = 2 \alpha \varphi - \sigma - \alpha \gamma - \varphi + \alpha \sigma + 3 \eta \rho + 3 \gamma \varphi + 3 \rho + 3 \gamma \sigma + \alpha^2 \eta - \alpha^2 \eta + \alpha \gamma^2 - \alpha^2 \varphi - 3 \eta^2 \varphi + 3 \gamma^2 \varphi - 3 \eta^2 \varphi + \gamma^2 \varphi
\]
\[
- 3 \eta^2 \sigma + \gamma^2 \sigma - 3 \gamma^2 \sigma + \gamma^2 \sigma - \alpha^2 \eta - \alpha^3 \eta \varphi + 7 \alpha^2 \eta - 7 \alpha^2 \eta \varphi + 3 \alpha^2 \eta \sigma + 3 \alpha^2 \eta \sigma + 3 \alpha^2 \eta \sigma + \alpha \gamma^2 \varphi
\]
\[
- \alpha^2 \eta \sigma + \alpha \gamma \sigma - 7 \alpha \eta \rho - 5 \alpha \rho - 2 \alpha \sigma - 6 \gamma \rho + 6 \gamma \rho - \alpha^2 \eta \sigma + 8 \alpha^2 \eta \rho - 3 \alpha^2 \eta \rho - \alpha^2 \eta \rho
\]
\[
+ 2 \alpha^2 \gamma \rho + 2 \alpha^2 \gamma \rho + 2 \alpha^2 \eta \sigma - 3 \alpha^2 \eta \sigma + \alpha \gamma^2 \sigma + 3 \gamma^2 \rho + 3 \gamma^2 \rho + 3 \gamma^2 \rho + 3 \gamma^2 \rho + 3 \gamma^2 \rho + 8 \alpha \gamma \sigma - 3 \alpha \gamma \sigma
\]
\[
- 6 \alpha \gamma \sigma - 4 \alpha \gamma \sigma - 3 \alpha \gamma \sigma - 6 \alpha \gamma \sigma + 3 \alpha \gamma \sigma + 3 \alpha \gamma \sigma + 3 \alpha \gamma \sigma + 10 \alpha \gamma \sigma.
\]

**Proof** See Appendix B.3. ■

Equation (43) describes markup fluctuations as a function of technology shocks. While (43) is more complex than the relationship we can obtain by expressing markup dynamics in terms of the other endogenous variables, it clearly reveals under what conditions the flexible price allocation is optimal. Corollary 3.1 lists the conditions when the flexible price allocation is optimal.
Corollary 3.1  Non-cooperative central banks under financial autarky implement the flexible price allocation whenever any of the four listed conditions below is satisfied:

- a) Under unitary import elasticity, $\eta = 1$ (including Cole-Obstfeld conditions $\sigma = \gamma = \eta = 1$)
- b) Under full home bias, $\alpha = 0$
- c) Under no home bias, $\alpha = 1$
- d) Under stable terms of trade, $\gamma \rightarrow \infty$ or $\eta \rightarrow \infty$.
- e) Under extreme risk-aversion, $\sigma \rightarrow \infty$

Proof  Under a), b), and c) in equation (43) we have $G_2 = 0$, which implies constant markups and flexible prices. With respect to d), as the export elasticity $\gamma$ increases, the numerator $F_3$ grows faster than $G_3$. More formally: $\lim_{\gamma \rightarrow \infty} \frac{G_3}{\gamma} \rightarrow -\alpha(1 - \alpha)(1 + \varphi)(\eta - 1), \lim_{\gamma \rightarrow \infty} \frac{F_3}{\gamma^3} \rightarrow \sigma + \varphi$. Thus, markups become more stable as the export elasticity increases and eventually become constant: $\lim_{\gamma \rightarrow \infty} \hat{\mu}_t = \lim_{\gamma \rightarrow \infty} \frac{G_3}{F_3} = 0$. We also know that as the import elasticity $\eta$ increases the numerator $F_3$ grows faster than $G_3$. More formally: $\lim_{\eta \rightarrow \infty} \frac{G_3}{\eta^2} \rightarrow -\alpha(1 - \alpha)(1 + \varphi), \lim_{\eta \rightarrow \infty} \frac{F_3}{\eta^3} \rightarrow (1 - \alpha)^3(\sigma + \varphi)$. Thus, markups become more stable as the import elasticity grows and eventually become constant: $\lim_{\eta \rightarrow \infty} \hat{\mu}_t = \lim_{\eta \rightarrow \infty} \frac{G_3}{F_3} = 0$. Finally, as risk-aversion $\sigma \rightarrow \infty$, the numerator $G_3$ remains constant, while the denominator $F_3 \rightarrow \infty$. Thus, markups become more stable or $\lim_{\sigma \rightarrow \infty} \hat{\mu}_t = \lim_{\sigma \rightarrow \infty} \frac{G_3}{F_3} = 0$.

In financial autarky, the intuition for case b) and d) in Corollary 3.1 is similar under cooperative and non-cooperative policy. In a closed economy (case b)), manipulation of the terms of trade does not generate any monopolistic rents as there are no exports. Thus, optimal cooperative and non-cooperative policy focuses on replicating the flexible price allocation. Regarding case d), as goods become more substitutable, national monopoly power at the export level is reduced, and in the limit the central bank’s ability to influence the terms of trade through monetary policy is eliminated. As a result, optimal monetary policy will focus on alleviating the distortion from nominal rigidities and the central bank will replicate the flexible price allocation. However, this is where the similarities between optimal cooperative and non-cooperative policies in financial autarky end.

Under extreme risk-aversion in part e), the flexible price allocation stabilizes consumption. Under cooperative policy in financial autarky, joint manipulation of the terms of trade allows policymakers to overcome their inability to share risk through international asset markets via the adjustment of...
international prices to stabilize consumption across countries. In the absence of coordination, deviation from flexible prices causes consumption and labor to fluctuate, which is infinitely costly under extreme risk-aversion. Thus, replication of the flexible price allocation is optimal for non-cooperative policymakers in financial autarky.

Also, the results on cooperative policies are no longer relevant for part a) and c) of Corollary 3.1. In a fully open economy and under unitary import elasticity, the flexible price allocation is optimal. In both cases the share of goods exported is constant and independent from technology shocks. While in a fully open economy the export share equals one, under unitary import elasticities the export share is equal to the degree of openness $\alpha$. We can formally show this by plugging (38) into (36), where we set $\eta = 1$, $1_{CM} = 0$, and $1_{CP} = 0$.

In Corollary 3.1 we see that a constant export share implies constant markups, which implies a relationship between the dynamics of the export share and optimal markups. Corollary 3.2 below formally establishes this relationship.

**Corollary 3.2** Under the optimal non-cooperative monetary policy in financial autarky, there is positive comovement between the markup and the export share.

**Proof** We can express markup dynamics in terms of export share dynamics using the results from Appendix B.3, equation (B.77):

$$\hat{\mu}_t = \frac{\alpha(\eta + \gamma - 1)}{(\eta(1 - \alpha) + \gamma - 1)^2 + \alpha(\eta(1 - \alpha) + \gamma - 1)} \hat{E}_{s,t}$$

For $\gamma \geq 1$, $\eta > 0$, $0 < \alpha < 1$, we have $\frac{\alpha(\eta + \gamma - 1)}{(\eta(1 - \alpha) + \gamma - 1)^2 + \alpha(\eta(1 - \alpha) + \gamma - 1)} > 0$. ■

Corollary 3.2 holds that when the export share goes up, the markup should increase. Since markups negatively comove with the output gap, the latter should also negatively correlate with the export share. In other words, monetary policy should be expansionary when the export share goes down, and contractionary when the share of goods exported goes up.

Why do optimal markups tend to move with the share of exports in production? In the steady state our model sets markups equal to one, which allows the implementation of the efficient cooperative steady state. However, under non-cooperative policy, the goal of the policymaker is to charge only domestic households markups equal to one. Since the country has monopolistic power with an elasticity of foreign demand equal to $\gamma$, the policymakers try to maximize their monopolistic rents by selling goods abroad with markups $\frac{\gamma}{\gamma - 1}$. But under the law of one price, the markup is the same
regardless of whether producers sell goods abroad or at home. Thus, the optimal markup is an average between one and $\frac{\gamma}{\gamma - 1}$. One can show that in the symmetric steady state the optimal non-cooperative markup is equal to $1 + \frac{\alpha}{\gamma - 1 + (1 - \alpha)\eta}$.

**Lemma 4** In the steady state, non-cooperative social planners maximize (34) subject to (36), (37), and (38), where the indicators for cooperative policy $1_{CP}$ and complete markets $1_{CM}$ are set to zero. The technology level $Z$ is equal to 1, as there are no technology shocks in steady state. The optimal markup for the social planner in the steady state is:

$$\mu = 1 + \frac{\alpha}{\gamma - 1 + (1 - \alpha)\eta}.$$  

**Proof** See Appendix C.3.

The optimal markup converges to one for the closed economy as $\alpha$ approaches zero, and converges to $\frac{\gamma}{\gamma - 1}$ for the fully open economy as $\alpha$ approaches one. In addition, the optimal steady state markup increases monotonically with $\alpha$.

**Lemma 5** The optimal markup for the non-cooperative social planner in the steady state monotonically increases with openness:

$$\frac{\partial \mu}{\partial \alpha} > 0.$$  

**Proof** $\frac{\partial \mu}{\partial \alpha} = \frac{\gamma + \eta - 1}{(\gamma - 1 + (1 - \alpha)\eta)^2} > 0$ for $0 < \alpha < 1$, $\gamma \geq 1$, and $\eta > 0$.

Lemmas 4 and 5 illustrate that steady state markups increase with openness $\alpha$, which is equal to the export share in steady state. In our model, we set the steady state markup to one. However, the optimal markup changes over the cycle since the export share changes. With a higher export share, the optimal monetary policy is to generate higher markups.

The closest relevant study was conducted by De Paoli (2009a), where she considered the small open economy as a limiting case of two large open economies. We differ from her paper in three dimensions. First, she uses Calvo pricing, while we utilize prices set one period in advance. Second, we differentiate between export and import elasticities. Finally, we analyze the solution for the global set of parameters, instead of focusing on a particular calibration. Overall, our results are consistent with De Paoli (2009a), and we confirm in the global parameter-space that the sign of the markup...
and the output gap depends on whether the import elasticity is greater or less than one. Our main contribution is the presentation of sufficient statistics for optimal policy, which holds whether we take trade elasticity estimates from the macro or trade literature.

Complete Markets and Non-Cooperative Policy

Optimal monetary policy for non-cooperative central banks under complete markets is the most complex of the four cases we consider. Two distortions drive equilibrium away from the first-best allocation: nominal rigidities and terms of trade externalities. The central bank faces a trade-off between replicating the flexible price allocation or extracting terms of trade rents. The main complicating factor in complete markets is that consumption dynamics differ from output dynamics. The decoupling of consumption from output makes complete markets different from financial autarky or the steady state. As a result, optimal monetary policy is more complicated. Before moving to the intuition, we state the principles for the optimal policy formally.

**Proposition 4** In complete markets, non-cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where the indicators for cooperative policy \( \mathbb{1}_{CP} \) and complete markets \( \mathbb{1}_{CM} \) are set to zero and one, correspondingly. The solution is given by

\[
\hat{\mu}_t = \frac{G_4}{F_4} \hat{Z}_t,
\]

where

\[
G_4 = \sigma \alpha (1 - \alpha) (1 + \psi) \left((1 - 2\eta)(\gamma \sigma - 1) \alpha + (\eta - 1)^2 + \sigma (\gamma - 1) + \eta^2 (\sigma - 1)\right),
\]

\[
F_4 = 4 \alpha \psi - \sigma - \psi + 3 \alpha \sigma + \eta \psi + \eta \sigma - 6 \alpha^2 \psi - 4 \alpha^3 \psi - 3 \alpha^2 \sigma + \alpha^3 \sigma + 2 \alpha \eta \sigma^2 + 3 \alpha^2 \eta \sigma^2 - \alpha \eta \sigma^2 + \alpha \gamma \sigma^2
\]

\[
- 5 \alpha \eta \psi - 4 \alpha \eta \sigma - 5 \alpha \eta \sigma - 5 \alpha \eta \sigma + 3 \alpha^2 \eta \sigma^2 + 3 \alpha^2 \eta \sigma^2 + 3 \alpha \eta \sigma^2 - 10 \alpha^2 \eta \psi - 10 \alpha^3 \eta \psi - 5 \alpha^3 \eta \psi + 6 \alpha^2 \eta \psi - 4 \alpha^3 \gamma \psi + \alpha^4 \gamma \psi
\]

\[
- 2 \alpha \psi^2 + 3 \alpha \eta \sigma - 3 \alpha^2 \eta \sigma - \alpha \sigma^2 + \alpha^2 \sigma + 2 \alpha \eta \sigma^2 - 2 \alpha \eta \sigma^2 - 2 \alpha^2 \eta \sigma^2 + 2 \alpha^2 \eta \sigma^2 - 2 \alpha \sigma^2 - \alpha^2 \psi^2 - 3 \alpha \eta^2 \psi^2 - 3 \alpha \eta^2 \psi^2 + 3 \alpha \eta^2 \psi^2 - 3 \alpha \eta^2 \psi^2 + 3 \alpha \eta^2 \psi^2 + 3 \alpha \eta^2 \psi^2
\]

\[
- \alpha \eta^2 \psi^2 - \alpha^2 \gamma \psi^2 + \alpha^2 \gamma \psi^2 + 3 \alpha \eta^2 \sigma^2 + 3 \alpha \eta^2 \sigma^2 - 6 \alpha \eta^2 \sigma^2 - 6 \alpha \eta^2 \sigma^2 + 6 \alpha \eta^2 \sigma^2 - 6 \alpha \eta^2 \sigma^2 + 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 - 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 - 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2 + 2 \alpha^2 \psi^2
\]

\[
+ 2 \alpha \eta \sigma^2 + 3 \alpha \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 - 3 \alpha^2 \eta \sigma^2 + 3 \alpha \eta^2 \psi^2 + 3 \alpha \eta^2 \psi^2 - 12 \alpha \eta^2 \psi^2 + 12 \alpha \eta^2 \psi^2 - 4 \alpha^2 \eta \sigma^2
\]

**Proof** See Appendix B.4.

Equation (45) describes markup dynamics as a function of technology shocks. While it is possible to express the markup in terms of output or other endogenous variables in a more transparent way, equation (45) shows the set of parameters under which the replication of flexible prices is optimal.
Corollary 4.1 lists the conditions under which replication of flexible price allocation is optimal.

**Corollary 4.1** Non-cooperative central banks under complete markets implement the flexible price allocation, whenever any of the four listed conditions below is satisfied:

- **a)** Under Cole-Obstfeld conditions \((\sigma = \gamma = \eta = 1)\).
- **b)** Under full home bias \((\alpha = 0)\).
- **c)** Under no home bias \((\alpha = 1)\).
- **d)** For an economy with stable terms of trade or whenever \(\gamma \to \infty\) or \(\eta \to \infty\).

**Proof** Under a), b), and c) in equation (45) we have \(G_4 = 0\), which implies constant markups and flexible prices. With respect to d), we know that as the export elasticity \(\gamma\) increases the numerator \(F_4\) grows faster than \(G_4\). More formally:
\[
\lim_{\gamma \to \infty} \frac{G_4}{\gamma} \to \alpha \sigma^2 \eta (1 - \alpha)(1 + \varphi), \quad \lim_{\gamma \to \infty} \frac{F_4}{\gamma^3} \to \alpha^2 \varphi \sigma^2.
\]
Thus, markups become more stable as the export elasticity increases:
\[
\lim_{\gamma \to \infty} \hat{\mu}_t = \lim_{\gamma \to \infty} \frac{G_4}{F_4} = 0.
\]
As the import elasticity \(\eta\) increases, the denominator \(F_4\) grows faster than the numerator \(G_4\). More formally:
\[
\lim_{\eta \to \infty} \frac{G_4}{\eta^2} \to -\sigma \alpha (1 - \alpha)(1 + \varphi)(1 + \sigma - 2\sigma \alpha), \quad \lim_{\eta \to \infty} \frac{F_4}{\eta^3} \to \alpha^2 (1 - \alpha)^3 \sigma^2 \varphi.
\]
Thus, markups become more stable as the import elasticity increases and eventually become constant:
\[
\lim_{\eta \to \infty} \hat{\mu}_t = \lim_{\eta \to \infty} \frac{G_4}{F_4} = 0.
\]

Similar to both cooperative and non-cooperative policies under financial autarky, price stability is the optimal non-cooperative policy in complete markets when the economy is fully closed (part b)), or when the terms of trade are always stable (part d)). In a closed economy, manipulation of the terms of trade does not generate any extra benefits because there are no exports, while for stable terms of trade the central bank cannot exploit the terms of trade externality to enhance domestic welfare.

Parts a) and c) are similar to non-cooperative policy under financial autarky. Under the Cole-Obstfeld calibration \((\sigma = \eta = \gamma = 1)\) or in a fully open economy \((\alpha = 1)\), mimicking the flexible price allocation is optimal. In both cases the export share remains constant. In financial autarky there is a positive relationship between the markup and the export share but unfortunately the relationship between the two is more complicated in complete markets. Using the solution for the markup from equation (45) and other endogenous variables from the Appendix B.4, we obtain the following relationship between the optimal markup and export share:

\[
\hat{\mu}_t = \frac{G_x}{F_x H_x} E_{s,t}, \quad (46)
\]
where

\[ G_x = \alpha \sigma (\sigma (1 - 2 \alpha) \eta^2 + \gamma \sigma \eta - (1 - \alpha) \eta (\sigma + 2) + 1 - \alpha). \]

\[ F_x = \alpha - 1 + \sigma (\gamma - \alpha \eta). \]

\[ H_x = (\eta (1 - \alpha) + \gamma - 1)(1 - \alpha)^2 + \alpha \sigma [\eta (\eta (1 - \alpha) + 2(2 \gamma - 1))(1 - \alpha) + \gamma (\gamma - 1)]. \]

In (46) the relationship between the optimal markup and the export share is no longer monotonic. The key intuitive difference between financial autarky and complete markets is that the former behaves more like the steady state in comparative statics. The monetary policy tradeoff under complete markets is to adjust the markup to extract higher terms of trade rents without generating too much volatility in consumption and hours worked. If consumption and labor become too volatile when the central bank attempts to exploit terms of trade externalities, it may be optimal for the policymaker to move toward the flexible price allocation and away from terms of trade manipulation. However, the more extracting higher terms of trade rents actually reduces the volatility of consumption and labor, the more optimal policy will deviate from replicating flexible prices.

As the parameter governing labor disutility is absent in (46), we focus now on the volatility of consumption. Consider a positive productivity shock. In general, under flexible prices the terms of trade should go down, consumption should go up, and the export share may increase or decrease. If the export share increases following a positive productivity shock, the terms of trade externality creates upward pressure on the markup, such that the policymaker will appreciate the terms of trade and the real exchange rate and decrease the level of consumption. Thus, a higher markup allows the policymaker to stabilize consumption and extract higher monopolistic rents, such that an increase in the markup is likely to be a dominant strategy when the export share is positively correlated with technology. On the other hand, if the export share declines after a positive productivity shock, the policymaker faces a different trade-off. In this case, the terms of trade externality creates downward pressure on the markup. However, a lower markup will depreciate the terms of trade and the real exchange rate, and increase consumption. Since consumption increases on impact, a further increase is undesirable when households are risk averse due to higher consumption volatility. Corollary 4.2 below formally establishes the relationship we have described.

**Corollary 4.2** Under complete markets and non-cooperative policy, markups are procyclical if the export share is procyclical and risk aversion (\( \sigma \)) is greater than one.

**Proof** See Appendix C.4. ■
Under what conditions might the export share decline after a positive productivity shock? First, this might happen if domestic consumption goes up too strongly in response to a shock. For example, under complete markets, consumption is more sensitive to the real exchange rate for lower values of risk aversion, and a positive productivity shock typically leads to a depreciation in the terms of trade and the real exchange rate and a rise in consumption. The sensitivity of consumption to the real exchange is inversely proportional to the coefficient of risk aversion. However, while a lower degree of risk aversion leads to higher consumption volatility, it also decreases the welfare loss from increased volatility. The second factor is home bias. The real exchange rate becomes more sensitive to the terms of trade as home bias increases. Thus, for a given fall in the terms of trade, the real exchange rate will depreciate and consumption will rise more strongly as home bias increases. The third factor is the relative magnitude of the import and export elasticities. If the import elasticity is significantly larger than the export elasticity, a decrease in the terms of trade will generate a rise in demand for the home good from domestic consumers that outstrips the rise in demand from abroad, leading to a decline in the export share.

To generate a disconnect between the export share and the markup, we need $\eta > \gamma$ in combination with a low degree of consumption home bias. In Figure 2 we deviate from the Cole-Obstfeld calibration by increasing $\eta$ and considering both a low and high degree of home bias.

![Figure 2: Response in Percent of the Markup and Export Share to a One Percent Increase in Technology Level](image)
Figure 2 plots the reaction of the markup and the export share for a particular combination of parameters following a one percent increase in technology. For example, the value of 0.2 on the vertical axis reflects that the markup or export share goes up by 0.2 percent in response to a one percent increase in the technology level. The subplot on the left corresponds to high home bias ($\alpha = 0.2$) while the subplot on the right corresponds to low home bias ($\alpha = 0.6$). Relative risk aversion and the elasticity of export substitution equal one in both subplots. When the import substitution elasticity $\eta$ is equal to one, we return to the Cole-Obstfeld case, where the export share is constant for any realization of the technology shock and the flexible price allocation is optimal. Thus, under the Cole-Obstfeld calibration, the markup and export share lines intersect at zero on the vertical axis and $\eta = 1$ on the horizontal axis as both the markup and the export share are constant in this case.

Under high home bias and high elasticity of import substitution $\eta$, the markup responds positively to a technology shock, while the export share declines. On the other hand, for low home bias, the markup and export share decline in response to a positive shock when home and foreign goods are substitutes ($\eta > 1$), while they both increase after a positive technology shock when home and foreign goods are complements ($\eta < 1$).

The comovement between the export share and markup is broken for high home bias and high $\eta$, but not for low home bias or low $\eta$. Why is this the case? With a positive technology shock, high home bias, and high $\eta$, the export share shrinks, and the terms of trade externality creates an incentive to reduce the markup. However, lowering the terms of trade in order to reduce markups has a strong effect on the real exchange rate and causes a substantial consumption increase when consumption is already high. This strong pressure from excess consumption volatility causes the policymaker to push markups in the opposite direction to stabilize consumption. Under low home bias, a reduction in the terms of trade has a small effect on the real exchange rate and causes only a small increase in consumption. In this case, markups can be countercyclical when the export share is countercyclical.

To summarize our findings on non-cooperative policy in complete markets, under the most realistic calibrations the optimal markup and the export share are procyclical. Monetary policy will generate negative output gaps in response to positive technology shocks by appreciating the real exchange rate and the terms of trade. We prove that when the export share is procyclical, optimal monetary policy requires the central bank engineer procyclical markups and countercyclical output gaps.

The closest relevant study was conducted by De Paoli (2009a,b), who considered the small open economy as a limiting case of two open economies. As in the previous section, we differ from her
paper in three dimensions. First, she uses Calvo pricing, while we utilize prices set one period in advance. Second, we differentiate between export and import elasticities. Finally, we analyze the solution for a global set of parameters, instead of focusing on a particular calibration.

For non-cooperative policy under complete markets, we find that differentiation between import and export elasticities plays a major role. Corollary 4.3 below shows that the markup and the export share are positively correlated.

**Corollary 4.3** If the import substitution elasticity $\eta$ is equal to the export substitution elasticity $\gamma$, and $\gamma > 1$, then the markup and the export share comove positively.

**Proof** See Appendix C.5. ■

Whenever relative risk aversion is greater than one, the comovement between the markup and the export share disappears only if the import elasticity is greater than the export elasticity.

### 4 Conclusion

There is a long tradition in macroeconomics of explaining the data by introducing distortions into perfectly competitive and efficient markets. In the closed economy this strategy has been fruitful and brought some immediate results in the form of the divine coincidence: by stabilizing inflation the central bank eliminates distortions arising from nominal rigidities, closes the output gap and removes price dispersion. Such a strategy has been less successful in the context of open economies, where imperfect cross-country risk-sharing and terms of trade externalities in addition to nominal rigidities prevent the divine coincidence.

Our aim here is to establish some simple principles for optimal cooperative and non-cooperative monetary policy in small open economies under complete markets and financial autarky. Relative to the literature, we do not consider the small open economy as the limiting case of two large economies, which allows us to differentiate between the export and import elasticity of substitution.

We are the first to consider cooperative optimal monetary policy for small open economies, which enables a clearer understanding of distortions in the absence of strategic interactions between countries. We find that mimicking the flexible price allocation is the optimal cooperative monetary policy under complete markets, which aligns with studies focused on two large economies. We also establish that under most realistic calibrations, cooperative optimal monetary policy under financial autarky deviates from price stability in favor of countercyclical markups, procyclical output gaps, and volatile terms of trade.
We then examine non-cooperative policy. Under financial autarky, markups set up by the monetary authority comove positively with the share of the goods exported regardless of the degree of risk aversion, home bias, or product substitutability across countries. Under complete markets, optimal non-cooperative monetary policy should generate procyclical markups and countercyclical output gaps whenever the export share is procyclical. This rule may be violated if consumption is strongly sensitive to monetary policy and is negatively correlated with the share of goods exported. In this case, central banks should restrain from lowering markups during a boom even if the export share falls, since responding to such movements with a lower markup might cause excess consumption volatility.

Across all four cases examined, the simple prescription for optimal monetary policy is to replicate flexible prices unless the export share is too volatile. If the export share is procyclical, then the optimal markup should also be procyclical. If the export share is countercyclical, then the optimal markup is countercyclical unless it leads to high consumption volatility.

To conclude, we find a simple monetary policy rule in the non-cooperative case. Policymakers should set markups to react to the export share unless a deviation of the markup from zero causes excess consumption volatility.
References


A Risk-Sharing

The household in country $i$ maximizes lifetime utility (1), subject to the household budget constraint (2) and the intertemporal asset constraint

$$
\mathbb{E} \left\{ \sum_{t=1}^{\infty} q_t D_{it} \right\} = 0. \quad (A.1)
$$

$D_{it}$ denotes nominal state-contingent portfolio payments in units of imported consumption baskets in period $t$, while $q_t$ is the price of the portfolio in period 0 (when all trading occurs), where $q_t$ does not depend on country $i$. The household in period 0 cares about the relative price of contingent claims across all states. The intertemporal asset constraint stipulates that all period 0 transactions must be balanced: payment for claims issued must equal payment for claims received. The household Lagrangian is:

$$
L_{it} = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \left\{ \frac{C_{it}^{1-\sigma} - N_{it}^{1+\varphi}}{1 - \sigma} + \lambda_{it} \left[ W_{it} N_{it} + P_{rif,it} D_{it} - P_{it} C_{it} \right] \right\} - \lambda_0 \sum_{t=1}^{\infty} \mathbb{E} \left\{ q_t D_{it} \right\}. \quad (A.2)
$$

Note that the Lagrange multiplier $\lambda_0$ is common across countries only under the assumption of ex-ante symmetry. The FOCs with respect to nominal state-contingent portfolio payments $D_{it}$ and consumption $C_{it}$ are:

$$
\frac{\partial L_{it}}{\partial D_{it}} = -\lambda_0 q_t + \beta^t \lambda_{it} \frac{P_{rif,it}}{P_{it}} = 0, \quad (A.3)
$$

$$
\frac{\partial L_{it}}{\partial C_{it}} = \beta^t C_{it}^{1-\sigma} - \beta^t \lambda_{it} = 0. \quad (A.4)
$$

Combining (A.3) and (A.4) gives the price of the portfolio $q_t$:

$$
q_t = \frac{\beta^t P_{rif,it}}{\lambda_0 P_{it}} C_{it}^{1-\sigma} = \frac{\beta^t C_{it}^{1-\sigma}}{\lambda_0 P_{it}}. \quad (A.5)
$$

Substituting (A.5) into (A.1), we can express the intertemporal asset constraint as

$$
\mathbb{E} \left\{ \sum_{t=1}^{\infty} \beta^t \frac{C_{it}^{1-\sigma}}{P_{it}} D_{it} \right\} = 0. \quad (A.6)
$$

which corresponds to equation (28) in the text.
Since the price of the portfolio $q_t$ and the Lagrange multiplier $\lambda_0$ do not depend on country $i$, the risk-sharing condition in complete markets is:

$$\frac{C_{it}^{-\sigma}}{P_{it}} = \frac{C_{jt}^{-\sigma}}{P_{jt}} = \frac{q_t \lambda_0}{\beta^t} \quad \forall \quad i, j$$

Therefore, $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{2}}$, where $A_t$ is known unconditionally as there is no aggregate uncertainty.

Since total portfolio payments globally add up to zero in each period,

$$\mathbb{E}\{D_{it}\} = \mathbb{E}\{\tilde{P}_{it} C_{it} - Y_{it} \tilde{P}_{H,it}\} = 0,$$

$$\Rightarrow \mathbb{E}\{\tilde{P}_{it} C_{it}\} = \mathbb{E}\{Y_{it} \tilde{P}_{H,it}\}. \quad (A.7)$$

Now substitute the expression $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{2}}$ for consumption in (A.7)

$$A_t = \frac{\mathbb{E}\{Y_{it} \tilde{P}_{H,it}\}}{\mathbb{E}\{\tilde{P}_{it}^{\frac{1}{2}}\}}, \quad (A.8)$$

and solve for $C_{it}$ using $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{2}}$:

$$C_{it} = \frac{\mathbb{E}\{Y_{it} \tilde{P}_{H,it}\}}{\mathbb{E}\{\tilde{P}_{it}^{\frac{1}{2}}\}} \tilde{P}_{it}^{-\frac{1}{2}}. \quad (A.9)$$
B  Propositions and Proofs

B.1  Proof of Proposition 1

Setting up the Lagrangian

In complete markets, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where \( \mathbb{1}_{CP} = 1 \) and \( \mathbb{1}_{CM} = 1 \). Thus, we can formulate a Lagrangian:

\[
\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \left[ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_1 (E_{t-1} c_t^{1-\sigma} - E_{t-1} N_t^{1+\varphi}) + \lambda_2 (1 - \alpha) \bar{p}_{H,t}^{-\gamma} C_t \bar{p}_t^\gamma + \alpha \bar{p}_{H,t}^{-\gamma} [C_t \bar{p}_t^\gamma - Z_t N_t] + \lambda_3 (1 - \alpha) \bar{p}_{H,t}^{-\gamma} + \alpha - \bar{p}_t^{-\gamma} + \lambda_4 \mathbb{E} \{ Z_t N_t \bar{p}_{H,t} \} - C_t \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \right].
\]  \( \text{(B.1)} \)

The first order conditions for the problem with respect to consumption \( C_t \), labor \( N_t \), the terms of trade \( \bar{p}_{H,t} \), and the real exchange rate \( \bar{p}_t \) are given below:

\[
\frac{\partial \mathcal{L}}{\partial C_t} = C_t^{-\sigma} + \lambda_1 (1 - \sigma) C_t^{-\sigma} + \lambda_2 (1 - \alpha) \bar{p}_{H,t}^{-\gamma} C_t \bar{p}_t^\gamma + \alpha E_{t-1} (\lambda_2 \bar{p}_{H,t}^{-\gamma} C_t \bar{p}_t^\gamma),
\]

\( = \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \bar{p}_t^{\gamma-1} - \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \bar{p}_t^{\gamma-1} = 0, \tag{B.2} \)

\[
\frac{\partial \mathcal{L}}{\partial N_t} = -N_t^{\varphi} + \lambda_1 (1 + \varphi) N_t^{\varphi} - \lambda_2 Z_t + \mathbb{E} \{ \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \bar{p}_t^{\gamma-1} \} Z_t = 0, \tag{B.3} \)

\[
\frac{\partial \mathcal{L}}{\partial \bar{p}_{H,t}} = -\eta (1 - \alpha) \lambda_2 \bar{p}_{H,t}^{-\gamma-1} C_t \bar{p}_t^\gamma - \gamma \alpha \lambda_2 \bar{p}_{H,t}^{-\gamma-1} E_{t-1} \{ C_t \bar{p}_t^\gamma \} + \lambda_3 (1 - \alpha) (1 - \eta) \bar{p}_{H,t}^{-\gamma} + \mathbb{E} \{ \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \bar{p}_t^{\gamma-1} \} Z_t = 0, \tag{B.4} \)

\[
\frac{\partial \mathcal{L}}{\partial \bar{p}_t} = (1 - \alpha) \eta \lambda_2 \bar{p}_{H,t}^{-\gamma-1} C_t \bar{p}_t^{\gamma-1} + \alpha \eta E_{t-1} (\lambda_2 \bar{p}_{H,t}^{-\gamma} C_t \bar{p}_t^{\gamma-1} - \bar{p}_t^{-\gamma} - \lambda_3 \bar{p}_t^{-\gamma} + \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} C_t \bar{p}_t^{\gamma-1} - \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} C_t \bar{p}_t^{\gamma-1} - \lambda_4 \mathbb{E} \{ \bar{p}_{H,t}^{-\gamma} \} \bar{p}_t^{\gamma-1} = 0. \tag{B.5} \)

The first order conditions (B.2)-(B.5), constraints (35)-(38), cooperation indicator \( \mathbb{1}_{CP} = 1 \), complete markets indicator \( \mathbb{1}_{CM} = 1 \), and exogenous shock dynamics (33) describe the full non-linear dynamics of the system. To obtain an analytical expression, we have to consider the behavior of the model in the steady state.

Steady State

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), goods and asset market clearing (36)-(38) in the steady state allows us to
show that $C = N = \hat{P}_H = \bar{P} = 1$. Substituting these values into the first order conditions (B.2)-(B.5) yields the following steady state relationships:

\begin{align}
\frac{\partial L}{\partial C} &= 1 + \lambda_1(1 - \sigma) + \lambda_2(1 - \alpha) + \alpha \lambda_2 - \lambda_4 = 0, \tag{B.6} \\
\frac{\partial L}{\partial N} &= -1 - \lambda_1(1 + \varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.7} \\
\frac{\partial L}{\partial \hat{P}_H} &= -\eta(1 - \alpha)\lambda_2 - \gamma \alpha \lambda_2 + \lambda_3(1 - \alpha)(1 - \eta) + \lambda_4 = 0, \tag{B.8} \\
\frac{\partial L}{\partial \hat{P}} &= (1 - \alpha)\eta \lambda_2 + \alpha \eta \lambda_2 - (1 - \eta)\lambda_3 - \lambda_4 \frac{1}{\sigma} - (1 - \frac{1}{\sigma})\lambda_4 = 0. \tag{B.9}
\end{align}

We solve the system (B.6)-(B.9) and obtain $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [0, \frac{1}{\gamma - 1}, \frac{\gamma - \eta}{(\gamma - 1)(\gamma - 1)}, \frac{\gamma}{\gamma - 1}]$.

**Log-Linearization**

We log-linearize the first order conditions (B.2)-(B.5) around the deterministic steady state and obtain:

\begin{align}
0 &= -\sigma(1 + \lambda_1(1 - \sigma))\dot{C} + \lambda_2(1 - \alpha)(\hat{\lambda}_2 - \eta \hat{P}_H + \eta \hat{P}_t) + \alpha \lambda_2 \eta \hat{P}_t \\
&\quad - \lambda_4(\hat{\lambda}_4 + \frac{1}{\sigma} \hat{P}_t) \tag{B.10} \\
0 &= -(1 + \lambda_1(1 + \varphi))\varphi \dot{N} - \lambda_2(\hat{\lambda}_2 + \hat{Z}_t) + \lambda_4(\hat{Z}_t + \hat{P}_H + \hat{P}_t) \tag{B.11} \\
0 &= -\eta(1 - \alpha)\lambda_2(\hat{\lambda}_2 - (\eta + 1)\hat{P}_H + \hat{C}_t + \eta \hat{P}_t) - \gamma \alpha \lambda_2(\hat{\lambda}_2 - (\gamma + 1)\hat{P}_H + \hat{N}_t) + \lambda_3(1 - \alpha)(1 - \eta)(\hat{\lambda}_3 - \eta \hat{P}_H) + \lambda_4(\hat{Z}_t + \hat{N}_t) \tag{B.12} \\
0 &= (1 - \alpha)\eta \lambda_2(\hat{\lambda}_2 - \eta \hat{P}_H + \hat{C}_t + (\eta - 1)\hat{P}_t) + \alpha \eta \lambda_2(\hat{C}_t + (\eta - 1)\hat{P}_t) + \lambda_3(\hat{\lambda}_3 - \eta \hat{P}_t) - \lambda_4 \frac{1}{\sigma}(\hat{\lambda}_4 + (\frac{1}{\sigma} - 1)\hat{P}_t + \hat{C}_t) + (1 - \frac{1}{\sigma})\frac{1}{\sigma} \lambda_4 \hat{P}_t \tag{B.13}
\end{align}

Now we log-linearize the constraints (35)-(38) after setting the value for the indicators $\mathbb{1}_{CP} = 1$, $\mathbb{1}_{CM} = 1$:

\begin{align}
0 &= -\hat{Z}_t - \hat{N}_t + (1 - \alpha)(-\eta \hat{P}_H + \hat{C}_t + \eta \hat{P}_t) - \alpha \gamma \hat{P}_H, \tag{B.14} \\
0 &= (1 - \alpha)\hat{P}_H - \hat{P}_t, \tag{B.15} \\
0 &= \hat{C}_t + \frac{1}{\sigma} \hat{P}_t. \tag{B.16}
\end{align}

We can express the system of linear equations consisting of (B.10)-(B.16) as

\[ A(\theta)X_t + b(\theta)Z_t = 0, \tag{B.17} \]
where $X_t = [\hat{C}_t, \hat{N}_t, \hat{P}_{H,t}, \hat{P}_t, \lambda_2, t, \lambda_3, t, \lambda_4, t]'$, $A$ is a $6 \times 6$ matrix, and $b$ is a $6 \times 1$ vector. After plugging in the values for $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, we can express coefficients in $A$ and $b$ as functions of the model parameters $\theta = [\sigma, \varphi, \alpha, \eta, \gamma]$. Finally, the endogenous variables $X_t$ can be expressed as a function of the parameter vector $\theta$ and the exogenous variable $Z_t$:

$$X_t = -A(\theta)^{-1}b(\theta)Z_t. \quad \text{(B.18)}$$

We can also express the markup as a function of parameters $\theta$ and the technology shock $Z_t$ by log-linearizing (25):

$$\hat{\mu}_t = \dot{Z}_t + \hat{P}_{H,t} - \hat{P}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \quad \text{(B.19)}$$

Plugging the solution from (B.18) into (B.19) yields the following expression for the log-linear markup:

$$\hat{\mu}_t = 0. \quad \text{(B.20)}$$

We refer the reader to Dmitriev and Hoddenbagh (2019) for proof that the flexible price allocation (they consider the flexible wage allocation but the algebra is identical) under complete markets coincides with the social planner allocation.

### B.2 Proof of Proposition 2

#### Setting up the Lagrangian

Under financial autarky, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 1$ and $\mathbb{1}_{CM} = 0$. The Lagrangian is:

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \left[ \frac{C_{t+1}^{1-\sigma}}{1-\sigma} - \frac{N_{t+1}^{1+\varphi}}{1+\varphi} + \lambda_1 (E_{t-1} C_{t+1}^{1-\sigma} - E_{t-1} N_{t+1}^{1+\varphi}) + \lambda_2 (1-\alpha) \hat{P}_{H,t}^\eta C_t \hat{P}_t^\eta + \alpha \hat{P}_{H,t}^\eta E_{t-1} [C_t \hat{P}_t^\eta] - Z_t N_t) + \lambda_3 ((1-\alpha) \hat{P}_{H,t}^\eta + \alpha \hat{P}_t^\eta) + \lambda_4 \left(Z_t N_t \hat{P}_{H,t} - C_t \hat{P}_t \right) \right]. \quad \text{(B.21)}$$
The first order conditions with respect to consumption $C_t$, labor $N_t$, terms of trade $\tilde{P}_{H,t}$, and real exchange rate $\tilde{P}_t$ are:

\[
\frac{\partial L}{\partial C_t} = C_t^{-\sigma} + \lambda_1(1 - \sigma)C_t^{-\sigma} + \lambda_2(1 - \alpha)\tilde{P}_{H,t}^{-\eta}\tilde{P}_t^{\eta} + \alpha E_{t-1}(\lambda_2, \tilde{P}_t^{-\gamma})\tilde{P}_t^{\eta} - \lambda_4, \tilde{P}_t = 0, \tag{B.22}
\]

\[
\frac{\partial L}{\partial N_t} = -N_t^\phi - \lambda_1(1 + \phi)N_t^\phi - \lambda_2, Z_t + \lambda_4, Z_t \tilde{P}_{H,t} = 0, \tag{B.23}
\]

\[
\frac{\partial L}{\partial \tilde{P}_{H,t}} = -\eta(1 - \alpha)\lambda_2, \tilde{P}_{H,t}^{-\eta-1}C_t \tilde{P}_t^{\eta} - \gamma\alpha\lambda_2, \tilde{P}_t^{-\gamma-1}E_{t-1}[C_t \tilde{P}_t^{\eta}]
+ \lambda_3, (1 - \alpha)(1 - \eta)\tilde{P}_{H,t}^{-\eta} + \lambda_4, Z_t N_t = 0, \tag{B.24}
\]

\[
\frac{\partial L}{\partial \tilde{P}_t} = (1 - \alpha)\eta\lambda_2, \tilde{P}_t^{-\eta}C_t \tilde{P}_t^{\eta-1} + \alpha\eta E_{t-1}(\lambda_2, \tilde{P}_t^{-\gamma})C_t \tilde{P}_t^{\eta-1} - (1 - \eta)\lambda_3, \tilde{P}_t^{-\eta} - \lambda_4, C_t = 0 \tag{B.25}
\]

The first order conditions (B.22)-(B.25), constraints (35)-(38), cooperation indicator $\mathbb{1}_{CP} = 1$, complete markets indicator $\mathbb{1}_{CM} = 0$, and exogenous shock dynamics (33) describe the full non-linear dynamics of the system. To obtain an analytical expression, we first need to solve for the steady state.

**Steady State. Financial Autarky. Cooperative.**

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), and goods and asset market clearing (36)-(38) in the steady state allows us to show that $C = N = \tilde{P}_H = \bar{\tilde{P}} = 1$. Substituting these values into the first order conditions (B.22)-(B.25) yields the following steady state relationships:

\[
\frac{\partial L}{\partial C} = 1 + \lambda_1(1 - \sigma) + \lambda_2(1 - \alpha) + \alpha\lambda_2 - \lambda_4 = 0, \tag{B.26}
\]

\[
\frac{\partial L}{\partial N} = -1 - \lambda_1(1 + \varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.27}
\]

\[
\frac{\partial L}{\partial \tilde{P}_H} = -\eta(1 - \alpha)\lambda_2 - \gamma\alpha\lambda_2 + \lambda_3(1 - \alpha)(1 - \eta) + \lambda_4 = 0, \tag{B.28}
\]

\[
\frac{\partial L}{\partial \tilde{P}} = (1 - \alpha)\eta\lambda_2 + \alpha\eta\lambda_2 - (1 - \eta)\lambda_3 - \lambda_4 = 0. \tag{B.29}
\]

We solve the system (B.26)-(B.29) and obtain $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [0, \frac{1}{\gamma-1}, \frac{\gamma-\eta}{(\eta-1)(\gamma-1)}, \frac{\gamma}{\gamma-1}]$. The steady state fully coincides with the steady state for cooperative policy under complete markets.
Log-Linearization

We log-linearize the first order conditions (B.22)-(B.25) around the deterministic steady state and obtain:

\[ 0 = -\sigma(1 + \lambda_1(1 - \sigma))\hat{C}_t + \lambda_2(1 - \alpha)(\hat{\lambda}_2,t - \eta\hat{P}_{H,t} + \eta\hat{P}_t) + \alpha\lambda_2\eta\hat{P}_t - \lambda_4(\hat{\lambda}_4,t + \hat{P}_t) \]  
(B.30)

\[ 0 = -(1 + \lambda_1(1 + \varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_2,t + \hat{Z}_t) + \lambda_4(\hat{\lambda}_4,t + \hat{Z}_t + \hat{P}_{H,t}) \]  
(B.31)

\[ 0 = -\eta(1 - \alpha)\lambda_2(\hat{\lambda}_2,t - (\eta + 1)\hat{P}_{H,t} + \hat{C}_t + \eta\hat{P}_t) + \gamma\alpha\lambda_2(\hat{\lambda}_2,t - (\gamma + 1)\hat{P}_{H,t}) + \lambda_3(1 - \alpha)(1 - \eta)(\hat{\lambda}_3,t - \eta\hat{P}_H,t) + \lambda_4(\hat{\lambda}_4,t + \hat{Z}_t + \hat{N}_t) \]  
(B.32)

\[ 0 = (1 - \alpha)\eta\lambda_2(\hat{\lambda}_2,t - \eta\hat{P}_{H,t} + \hat{C}_t + (\eta - 1)\hat{P}_t) + \alpha\eta\lambda_2(\hat{C}_t + (\eta - 1)\hat{P}_t) - (1 - \eta)\lambda_3(\hat{\lambda}_3,t - \eta\hat{P}_t) - \lambda_4(\hat{\lambda}_4,t + \hat{C}_t) \]  
(B.33)

Now we log-linearize the constraints (35)-(38) after setting the value for the indicators \(1_{CP} = 1, 1_{CM} = 0\):

\[ 0 = -\hat{Z}_t - \hat{N}_t + (1 - \alpha)(-\eta\hat{P}_{H,t} + \hat{C}_t + \eta\hat{P}_t) - \alpha\gamma\hat{P}_{H,t} \]  
(B.34)

\[ 0 = (1 - \alpha)\hat{P}_{H,t} - \hat{P}_t \]  
(B.35)

\[ 0 = -\hat{C}_t - \hat{P}_t + \hat{Z}_t + \hat{N}_t + \hat{P}_{H,t} \]  
(B.36)

We can express the system of linear equations consisting of (B.30)-(B.36) as

\[ A(\theta)X_t + b(\theta)Z_t = 0, \]  
(B.37)

where \(X_t = [\hat{C}_t, \hat{N}_t, \hat{P}_{H,t}, \hat{P}_t, \hat{\lambda}_2,t, \hat{\lambda}_3,t, \hat{\lambda}_4,t]^{\prime}\), \(A\) is a 6 \times 6 matrix, and \(b\) is a 6 \times 1 vector. After plugging in the values for \([\lambda_1, \lambda_2, \lambda_3, \lambda_4]\), we can express coefficients in \(A\) and \(b\) as functions of the model parameters \(\theta = [\sigma, \varphi, \alpha, \eta, \gamma]\). Finally, the endogenous variables \(X_t\) can be expressed as a function of the parameter vector \(\theta\) and the exogenous variable \(Z_t\):

\[ X_t = -A(\theta)^{-1}b(\theta)Z_t. \]  
(B.38)

We can also express the markup as function of parameters \(\theta\) and the technology shock \(Z_t\) by log-linearizing (25):
\[ \mu_t = \hat{Z}_t + \hat{P}_{H,t} - \hat{P}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \]  
(B.39)

**Analytical Expression**

Plugging the solution from (B.38) into (B.39) yields the following expression for the log-linear markup:

\[ \hat{\mu}_t = \frac{G_2(\theta)}{F_2(\theta)} \hat{Z}_t, \]  
(B.40)

where

\[ G_2(\theta) = -\alpha(1 + \varphi)((1 - \alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)). \]  
(B.41)

\[ F_2(\theta) = \varphi + \sigma + \alpha\eta + \alpha\gamma - 2\eta\sigma + \eta^2\sigma - 2\gamma\sigma - \alpha^2\eta - 2\alpha\eta\sigma - 2\alpha\gamma\sigma + 2\eta\gamma\sigma + 2\alpha\eta\sigma - 2\alpha^2\varphi - 2\alpha^2\eta\varphi - 2\alpha^2\gamma\varphi. \]  
(B.42)

We can express output \( \hat{Y}_t = \hat{N}_t + \hat{Z}_t \) as a function of the technology shock using equation (B.38):

\[ \hat{Y}_t = \frac{G_3,\gamma}{F_3,\gamma} \hat{Z}_t, \]  
(B.43)

where

\[ G_3,\gamma = \varphi - 2\eta - 2\gamma - 2\alpha + 4\alpha\eta + 2\alpha\gamma + 2\eta\gamma - 2\alpha\eta\varphi - 2\alpha\gamma\varphi - 2\alpha^2\eta - 2\alpha^2\varphi + \eta^2\varphi + \gamma^2\varphi + \alpha^2\eta^2\varphi + 2\alpha\eta\gamma + 4\alpha\eta\varphi + 2\alpha\gamma\varphi + 2\eta\gamma\varphi - 2\alpha^2\eta\varphi - 2\alpha^2\gamma\varphi + 1, \]  
(B.44)

\[ F_3,\gamma = \varphi + \sigma + \alpha\eta + \alpha\gamma - 2\alpha\varphi - 2\eta\varphi - 2\gamma\varphi - 2\eta\gamma - 2\alpha\eta\varphi + \alpha^2\eta\varphi + \gamma^2\varphi + \alpha^2\gamma^2\varphi + \alpha^2\eta^2\varphi \gamma + \alpha^2\eta\gamma^2\varphi + 4\alpha\eta\gamma + 2\alpha\eta\varphi + 2\alpha\gamma\varphi + 2\eta\gamma\varphi - 2\alpha^2\eta\varphi - 2\alpha^2\gamma\varphi - 2\alpha^2\eta\varphi - 2\alpha^2\gamma\varphi - 2\alpha^2\eta\gamma\varphi. \]  
(B.45)

Given the solutions for the markup and output in terms of technology shocks allows us to obtain their ratio and express it below:

\[ \hat{\mu}_t = -\alpha \frac{(1 - \alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)}{(1 - \alpha)^2(\eta - 1)^2 + (\gamma - 1)^2 + 2\eta\gamma(1 - \alpha) + 2\alpha\gamma - 1} \hat{Y}_t \]  
(B.46)
B.3 Proof of Proposition 3

Setting up the Lagrangian

Under financial autarky, non-cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $1_{CP} = 0$ and $1_{CM} = 0$. The Lagrangian is:

$$L = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \left[ \frac{C_t^{1-\sigma}}{1 - \sigma} - \frac{N_t^{1+\varphi}}{1 + \varphi} + \lambda_1 (E_{t-1} C_t^{1-\sigma} - E_{t-1} N_t^{1+\varphi}) + \lambda_2, t \left( (1 - \alpha) \tilde{P}_{H,t} - \eta H,t \tilde{P}_t \eta + \alpha - \tilde{P}_t^{1-\eta} \right) + \lambda_4, t \left( Z_t N_t - C_t \tilde{P}_t \right) \right].$$ (B.47)

The first order conditions with respect to consumption $C_t$, labor $N_t$, the terms of trade $\tilde{P}_{H,t}$, and the real exchange rate $\tilde{P}_t$ are given below:

$$\frac{\partial L}{\partial C_t} = C_t^{1-\sigma} + \lambda_1 (1 - \sigma) C_t^{1-\sigma} + \lambda_2, t (1 - \alpha) \tilde{P}_{H,t} - \eta \tilde{P}_t^\gamma \tilde{P}_t - \lambda_4, t \tilde{P}_t = 0, \quad \text{(B.48)}$$

$$\frac{\partial L}{\partial N_t} = -N_t^\varphi - \lambda_1 (1 + \varphi) N_t^\varphi - \lambda_2, t Z_t + \lambda_4, t Z_t \tilde{P}_{H,t} = 0. \quad \text{(B.49)}$$

$$\frac{\partial L}{\partial \tilde{P}_{H,t}} = - \eta (1 - \alpha) \lambda_2, t \tilde{P}_{H,t}^{1-\eta} C_t \tilde{P}_t - \gamma \alpha \lambda_2, t \tilde{P}_{H,t}^{1-\gamma} C_t + \lambda_3, t (1 - \alpha) (1 - \eta) \tilde{P}_{H,t}^{1-\eta} + \lambda_4, t Z_t N_t = 0, \quad \text{(B.50)}$$

$$\frac{\partial L}{\partial \tilde{P}_t} = (1 - \alpha) \eta \lambda_2, t \tilde{P}_{H,t}^{1-\eta} C_t \tilde{P}_t^{1-\eta} - (1 - \eta) \lambda_3, t \tilde{P}_t^{1-\eta} - \lambda_4, t C_t = 0. \quad \text{(B.51)}$$

The first order conditions (B.48)-(B.51), constraints (35)-(38), cooperation indicator $1_{CP} = 0$, complete markets indicator $1_{CM} = 0$, and exogenous shock dynamics (33) describe the full non-linear dynamics of the system. To obtain an analytical expression, we must first consider the behavior of the model in the steady state.

Steady State

Solving for the optimal pricing constraint (35), the relationship between the terms of trade and the real exchange rate (37), goods and asset market clearing (36)-(38) in the steady state shows that $C = N = \tilde{P}_H = \tilde{P} = 1$. Substituting these values into the first order conditions (B.48)-(B.51) yields the
following steady state relationships:

\[
\frac{\partial L}{\partial C} = 1 + \lambda_1(1 - \sigma) + \lambda_2(1 - \alpha) - \lambda_4 = 0, \tag{B.52}
\]

\[
\frac{\partial L}{\partial N} = -1 - \lambda_1(1 + \varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.53}
\]

\[
\frac{\partial L}{\partial \tilde{P}_H} = -\eta(1 - \alpha)\lambda_2 - \gamma\alpha, \lambda_2 + \lambda_3(1 - \alpha)(1 - \eta) + \lambda_4 = 0, \tag{B.54}
\]

\[
\frac{\partial L}{\partial \tilde{P}} = (1 - \alpha)\eta\lambda_2 - (1 - \eta)\lambda_3 - \lambda_4 = 0. \tag{B.55}
\]

We solve the system (B.52)-(B.55) and obtain

\[
\lambda_1 = \frac{\alpha}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\sigma + \eta\sigma - \alpha\eta\varphi - \alpha\eta\sigma}. \tag{B.56}
\]

\[
\lambda_2 = \frac{\varphi + \sigma}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\sigma + \eta\sigma - \alpha\eta\varphi - \alpha\eta\sigma}. \tag{B.57}
\]

\[
\lambda_3 = \frac{\gamma(\varphi + \sigma)}{(\eta - 1)(\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\sigma + \eta\sigma - \alpha\eta\varphi - \alpha\eta\sigma)}, \tag{B.58}
\]

\[
\lambda_4 = \frac{(\varphi + \sigma)(\eta + \gamma - \alpha\eta)}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\sigma + \eta\sigma - \alpha\eta\varphi - \alpha\eta\sigma}. \tag{B.59}
\]

**Log-Linearization.**

We log-linearize the first order conditions (B.48)-(B.51) around the deterministic steady state and obtain:

\[
0 = -\sigma(1 + \lambda_1(1 - \sigma))\hat{C}_t + \lambda_2(1 - \alpha)(\hat{\lambda}_{2,t} - \eta\hat{P}_{H,t} + \eta\hat{P}_t) - \lambda_4(\hat{\lambda}_{4,t} + \hat{\tilde{P}}_t) \tag{B.60}
\]

\[
0 = -(1 + \lambda_1(1 + \varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{\tilde{P}}_{H,t}) \tag{B.61}
\]

\[
0 = -\eta(1 - \alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta + 1)\hat{P}_{H,t} + \hat{C}_t + \eta\hat{P}_t) - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma + 1)\hat{P}_{H,t}) + \lambda_3(1 - \alpha)(1 - \eta)(\hat{\lambda}_{3,t} - \eta\hat{P}_{H,t}) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{N}_t) \tag{B.62}
\]

\[
0 = (1 - \alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{P}_{H,t} + \hat{C}_t + (\eta - 1)\hat{P}_t) - (1 - \eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{P}_t) - \lambda_4(\hat{\lambda}_{4,t} + \hat{C}_t) \tag{B.63}
\]

Now we log-linearize the constraints (35)-(38) after setting the value for the indicators $1_{CP} = 0$, \ldots
CM = 0:

\[ 0 = -\hat{Z}_t - \hat{N}_t + (1 - \alpha)(-\eta \hat{P}_{H,t} + \hat{C}_t + \eta \hat{P}_t) - \alpha \gamma \hat{P}_{H,t} \]  
\[ (B.64) \]

\[ 0 = (1 - \alpha) \hat{P}_{H,t} - \hat{P}_t \]  
\[ (B.65) \]

\[ 0 = -\hat{C}_t - \hat{B}_t \hat{N}_t + \hat{Z}_t + \hat{P}_{H,t} \]  
\[ (B.66) \]

We can express the system of linear equations consisting of (B.60)-(B.66) as

\[ A(\theta)X_t + b(\theta)Z_t = 0, \]  
\[ (B.67) \]

where \( X_t = [\hat{C}_t, \hat{N}_t, \hat{P}_{H,t}, \hat{P}_t, \hat{\lambda}_{2,t}, \hat{\lambda}_{3,t}, \hat{\lambda}_{4,t}]' \), \( A \) is a 6 \times 6 matrix, and \( b \) is a 6 \times 1 vector. After plugging in the values for \( [\lambda_1, \lambda_2, \lambda_3, \lambda_4] \), we can express the coefficients in \( A \) and \( b \) as functions of the model parameters \( \theta = [\sigma, \varphi, \alpha, \eta, \gamma] \). As a result, we can express the endogenous variables \( X_t \) as a function of the parameter vector \( \theta \) and the exogenous variable \( Z_t \):

\[ X_t = -A(\theta)^{-1}b(\theta)Z_t. \]  
\[ (B.68) \]

We can express the markup as a function of parameters \( \theta \) and the technology shock \( Z_t \) by log-linearizing (25):

\[ \mu_t = \hat{Z}_t + \hat{P}_{H,t} - \hat{P}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \]  
\[ (B.69) \]

**Analytical Expression**

Plugging the solution from (B.68) into (B.69) yields the following expression for the log-linear markup:

\[ \tilde{\mu}_t = \frac{G_3}{F_3} \hat{Z}_t. \]  
\[ (B.70) \]
where

\[ G_3 = -\alpha(1 - \alpha)(1 + \varphi)(\eta - 1)(\eta - 1 + \gamma). \]  

(B.71)

\[ F_3 = 2\alpha\varphi - \sigma - \alpha\varphi + \alpha\varphi + 3\eta\varphi + 3\eta\varphi + 3\eta\varphi + 3\eta\varphi + \alpha^2\eta - \alpha^2\eta + \alpha^2\varphi - 3\eta^2\varphi + \eta^2\varphi \]

\[ - 3\gamma^2 + \varphi^2 - 3\eta^2 + 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - \gamma^2 + 7\alpha^2\eta^2 + 3\alpha^2\eta^2 + 2\alpha^2\eta^2 - \alpha^2\eta^2 \]

\[ - 5\alpha^2\eta^2 + 3\alpha^2\eta^2 + \alpha^2\eta^2 + \alpha^2\eta^2 + \alpha^2\eta^2 + 2\alpha^2\eta^2 - \alpha^2\eta^2 \]

\[ + 8\alpha^2\eta^2 + 5\alpha^2\eta^2 - 3\alpha^2\eta^2 + \alpha^2\eta^2 + 2\alpha^2\eta^2 + 2\alpha^2\eta^2 + 7\alpha^2\eta^2 \]

\[ + 3\eta^2 + 3\eta^2 + 8\alpha^2\eta^2 - 3\alpha^2\eta^2 - 6\alpha^2\eta^2 - 4\alpha^2\eta^2 - 3\alpha^2\eta^2 - 3\alpha^2\eta^2 - 2\alpha^2\eta^2 + 3\alpha^2\eta^2 \]

\[ + 3\alpha^2\eta^2 - 10\alpha\eta\gamma. \]  

(B.72)

We log-linearize the export share (17) and use the production function (18) to obtain:

\[ \hat{E}_{s,t} = -\gamma \hat{P}_{H,t} - \hat{Z}_t - \hat{N}_t \]  

(B.73)

Now substitute the solution for \( \hat{N}_t \) and \( \hat{P}_{H,t} \) from (B.68) into (B.73) to get the log-linear export share:

\[ \hat{E}_{s,t} = \frac{G_{3,E}}{F_{3,E}} \hat{Z}_t, \]  

(B.74)

where

\[ G_{3,E} = -(2\alpha + 3\eta + 2\gamma - \varphi - 7\alpha\eta - 3\alpha\gamma - 4\gamma\eta + 2\alpha\varphi + 3\eta\varphi + 2\gamma\varphi + 8\alpha\eta^2 + 5\alpha^2\eta - 3\alpha\eta^3 - \alpha^2\eta + \alpha\gamma^2 + \alpha^2\gamma + \eta\gamma^2 \]

\[ + 2\eta^2\gamma - \alpha\gamma^2 - 3\eta^2\varphi + \eta\gamma^2 - \gamma^2 - \gamma^2 - 3\eta^2 + 3\gamma^2 - 7\alpha^2\eta^2 + 3\alpha^2\eta^2 + 2\alpha^2\eta^2 - \alpha^2\eta^2 + 2\alpha^2\gamma - 7\alpha^2\eta^2 \]

\[ + 3\alpha^2\eta^2 + 2\alpha^2\eta^2 - \alpha^2\eta^2 + 7\alpha\gamma - 7\alpha\gamma - 7\alpha\gamma - 3\alpha\gamma - 4\gamma\varphi - 3\alpha\gamma - 3\alpha\gamma + 4\alpha\gamma - 3\alpha\gamma + 3\alpha\gamma - 3\alpha\gamma \]

\[ - \alpha^2\gamma + \alpha\gamma^2 + \alpha^2\gamma + \gamma^2 + 2\eta^2\gamma - \alpha\gamma^2 - \gamma^2 - \gamma^2 - \gamma^2 - \gamma^2 - \gamma^2 - \alpha^2\eta + \alpha\gamma^2 + \alpha^2\eta + \gamma^2 \]

\[ - 3\eta^2 - 3\eta^2 + 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 - 3\gamma^2 \]

\[ + 2\alpha^2\varphi + \alpha^2\varphi + 3\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi \]

\[ + 2\alpha^2\varphi + \alpha^2\varphi + 3\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi \]

\[ + 2\alpha^2\varphi + \alpha^2\varphi + 3\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi \]

\[ + 2\alpha^2\varphi + \alpha^2\varphi + 3\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi \]

\[ + 2\alpha^2\varphi + \alpha^2\varphi + 3\alpha^2\varphi + 2\alpha^2\varphi + 2\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi + 3\alpha^2\varphi \]

\[ - 6\alpha^2\gamma - 4\alpha^2\gamma - 3\alpha^2\gamma - 6\alpha^2\gamma - 2\alpha^2\gamma - 3\alpha^2\gamma + 3\alpha^2\gamma - 3\alpha^2\gamma + 3\alpha^2\gamma - 3\alpha^2\gamma + 3\alpha^2\gamma. \]  

(B.75)

Then we can express the markup as a function of the export share using the formula:

\[ \frac{\mu_t}{E_{s,t}} = \frac{G_3}{F_3} = \frac{\alpha(\eta + \gamma - 1)}{(\eta(1 - \alpha) + \gamma - 1)^2 + \alpha(\eta(1 - \alpha) + \gamma - 1)}. \]  

(B.77)
B.4 Proof of Proposition 4

Lagrangian

Under complete markets, non-cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 0$ and $\mathbb{1}_{CM} = 1$. The Lagrangian is:

$$
L = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \left[ \frac{C_{t-1}^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} + \lambda_1(E_{t-1}C_{t-1}^{1-\sigma} - E_{t-1}N_t^{1+\psi}) + \lambda_{2,t}((1-\alpha)(1-\eta)C_{t-1}^{1-\sigma} + \alpha C_{t-1}^{1-\sigma} - Z_tN_t) + \lambda_{3,t}((1-\alpha)\tilde{P}_{h,t}^{1-\eta} + \alpha - \tilde{P}_{t}^{1-\eta}) + \lambda_{4,t}\left( E_{t-1}\{Z_tN_t\tilde{P}_{h,t}\} - C_t\tilde{P}_{t}^{\frac{1}{\gamma}} E_{t-1}[\tilde{P}_{t}^{\frac{1}{\gamma}}] \right) \right].
$$

The first order conditions for the problem with respect to consumption $C_t$, labor $N_t$, the terms of trade $\tilde{P}_{H,t}$, and the real exchange rate $\tilde{P}_{t}$ are:

$$
\frac{\partial L}{\partial C_t} = C_t^{1-\sigma} + \lambda_1(1-\sigma)C_t^{1-\sigma} + \lambda_{2,t}(1-\alpha)\tilde{P}_{h,t}^{\gamma-1}\tilde{P}_{t}^{\gamma} - \lambda_{4,t}\tilde{P}_{t}^{\frac{1}{\gamma}} E_{t-1}[\tilde{P}_{t}^{\frac{1}{\gamma}}],
$$

$$
\frac{\partial L}{\partial N_t} = -N_t^{\psi} - \lambda_1(1+\psi)N_t^{\psi} - \lambda_{2,t}Z_t + E(\lambda_{4,t})Z_t\tilde{P}_{h,t},
$$

$$
\frac{\partial L}{\partial \tilde{P}_{H,t}} = -\eta(1-\alpha)\lambda_{2,t}\tilde{P}_{h,t}^{\gamma-1}C_t\tilde{P}_{t}^{\gamma} - \gamma\alpha\lambda_{2,t}\tilde{P}_{h,t}^{\gamma-1}C_F + \lambda_{3,t}(1-\alpha)(1-\eta)\tilde{P}_{H,t}^{\gamma-1}

+ E(\lambda_{4,t})Z_tN_t,
$$

$$
\frac{\partial L}{\partial \tilde{P}_{t}} = (1-\alpha)\eta\lambda_{2,t}\tilde{P}_{h,t}^{\gamma-1}C_t\tilde{P}_{t}^{\gamma-1} - (1-\eta)\lambda_{3,t}\tilde{P}_{t}^{\gamma-1} - \lambda_{4,t}\frac{1}{\sigma}\tilde{P}_{t}^{\gamma-1}C_tE_{t-1}[\tilde{P}_{t}^{\gamma-1}]

- (1-\frac{1}{\sigma})E\left[\lambda_{4,t}C_t\tilde{P}_{t}^{\gamma-1}\right]\tilde{P}_{t}^{\frac{1}{\gamma-1}}.
$$

These first order conditions (B.79)-(B.82), constraints (35)-(38), cooperation indicator $\mathbb{1}_{CP} = 0$, complete markets indicator $\mathbb{1}_{CM} = 1$, and exogenous shock dynamics (33) describe the full non-linear dynamics of the system. To obtain an analytical expression, we must first consider the behavior of the model in the steady state.

**Steady State.**

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), and goods and asset market clearing (36)-(38) in the steady state reveals that $C = N = \tilde{P}_{H} = \tilde{P} = 1$. Substituting these values into the first order conditions (B.48)-(B.51) yields the
following steady state relationships:

\[
\frac{\partial L}{\partial C} = 1 + \lambda_1(1 - \sigma) + \lambda_2(1 - \alpha) - \lambda_4 = 0, \quad (B.83)
\]
\[
\frac{\partial L}{\partial N} = -1 - \lambda_1(1 + \varphi) - \lambda_2 + \lambda_4 = 0, \quad (B.84)
\]
\[
\frac{\partial L}{\partial \hat{P}_H} = -\eta(1 - \alpha)\lambda_2 - \gamma\alpha\lambda_2 + \lambda_3(1 - \alpha)(1 - \eta) + \lambda_4, \quad (B.85)
\]
\[
\frac{\partial L}{\partial \bar{P}} = (1 - \alpha)\eta\lambda_2 - (1 - \eta)\lambda_3 - \lambda_4 \frac{1}{\sigma} - (1 - \frac{1}{\sigma})\lambda_4. \quad (B.86)
\]

We solve the system (B.83)-(B.86) and obtain

\[
\lambda_1 = -\frac{\alpha}{\alpha - \varphi - \sigma + \alpha\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\varphi - \alpha\eta\sigma}, \quad (B.87)
\]
\[
\lambda_2 = \frac{\varphi + \sigma}{\alpha - \varphi - \sigma + \alpha\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\varphi - \alpha\eta\sigma}, \quad (B.88)
\]
\[
\lambda_3 = \frac{\gamma(\varphi + \sigma)}{(\eta - 1)(\alpha - \varphi - \sigma + \alpha\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\varphi - \alpha\eta\sigma)}, \quad (B.89)
\]
\[
\lambda_4 = \frac{(\varphi + \sigma)(\eta + \gamma - \alpha\eta)}{\alpha - \varphi - \sigma + \alpha\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\varphi - \alpha\eta\sigma}. \quad (B.90)
\]

The steady state fully coincides with the steady state for cooperative policy under financial autarky.

**Log-Linearization**

We log-linearize the first order conditions (B.48)-(B.51) around the deterministic steady state and obtain:

\[
0 = -\sigma(1 + \lambda_1(1 - \sigma))\hat{C}_t + \lambda_2(1 - \alpha)(\hat{\lambda}_{2,t} - \eta\hat{P}_{H,t} + \eta\hat{P}_t) - \lambda_4(\hat{\lambda}_{4,t} + \frac{1}{\sigma}\hat{P}_t), \quad (B.91)
\]
\[
0 = -(1 + \lambda_1(1 + \varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{Z}_t + \hat{P}_{H,t}), \quad (B.92)
\]
\[
0 = -\eta(1 - \alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta + 1)\hat{P}_{H,t} + \hat{C}_t + \eta\hat{P}_t) - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma + 1)\hat{P}_{H,t}) + \lambda_3(1 - \alpha)(1 - \eta)(\hat{\lambda}_{3,t} - \eta\hat{P}_{H,t}) + \lambda_4(\hat{Z}_t + \hat{N}_t), \quad (B.93)
\]
\[
0 = (1 - \alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{P}_{H,t} + \hat{C}_t + (\eta - 1)\hat{P}_t) - (1 - \eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{P}_t) - \lambda_4\frac{1}{\sigma}(\hat{\lambda}_{4,t} + \frac{1}{\sigma} - 1)\hat{P}_t + \hat{C}_t) + (1 - \frac{1}{\sigma})\frac{1}{\sigma}\lambda_4\hat{P}_t. \quad (B.94)
\]

Now we log-linearize the constraints (35)-(38) after setting the value for the indicators \(1_{CP} = 0\).
\( 1_{CM} = 0:\)

\[
0 = -Z_t - \hat{N}_t + (1 - \alpha)(-\eta \hat{P}_{H,t} + \hat{C}_t + \eta \hat{P}_t) - \alpha \gamma \hat{P}_{H,t}, \tag{B.95}
\]

\[
0 = (1 - \alpha) \hat{P}_{H,t} - \hat{P}_t, \tag{B.96}
\]

\[
0 = \hat{C}_t + \frac{1}{\sigma} \hat{P}_t. \tag{B.97}
\]

We can express the system of linear equations consisting of (B.91)-(B.97) as

\[
A(\theta)X_t + b(\theta)Z_t = 0, \tag{B.98}
\]

where \( X_t = [\hat{C}_t, \hat{N}_t, \hat{P}_{H,t}, \hat{P}_t, \lambda_{2,t}, \lambda_{3,t}, \lambda_{4,t}]' \), \( A \) is a 6 x 6 matrix, and \( b \) is a 6 x 1 vector. After plugging in the values for \( [\lambda_1, \lambda_2, \lambda_3, \lambda_4] \), we can express the coefficients in \( A \) and \( b \) as functions of the model parameters \( \theta = [\sigma, \varphi, \alpha, \eta, \gamma] \). As a result, the endogenous variables \( X_t \) can be expressed as a function of the parameter vector \( \theta \) and the exogenous variable \( Z_t \):

\[
X_t = -A(\theta)^{-1}b(\theta)Z_t. \tag{B.99}
\]

We can express the markup as a function of the parameters \( \theta \) and the technology shock \( Z_t \) by log-linearizing (25):

\[
\mu_t = \hat{Z}_t - \hat{C}_t + \hat{P}_{H,t} - \hat{P}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \tag{B.100}
\]

**Analytical Expression**

Plugging the solution from (B.99) into (B.100) allows us to obtain the expression for markup:

\[
\hat{\mu}_t = \frac{G_4}{F_4} \hat{Z}_t. \tag{B.101}
\]
We can log-linearize the export share (17) and use the production function (18) to obtain:

\[
\hat{E}_{s,t} = -\gamma \hat{P}_{H,t} - \hat{Z}_t - \hat{N}_t
\]  

(B.102)

We substitute the solution for \( \hat{N}_t \) and \( \hat{P}_{H,t} \) from (B.99) into (B.102) to get the log-linear export share:

\[
\hat{E}_{s,t} = \frac{G_{4,E}}{F_{4,E}} \hat{Z}_t.
\]  

(B.103)

We do not provide the full expressions for \( G_{4,E} \) and \( F_{4,E} \) as they are enormous and provide little value by themselves. Instead, we express the log-linear markup as a function of the log-linear export share using the formula:

\[
\frac{\hat{\mu}_t}{\hat{E}_{s,t}} = \frac{G_h}{F_h H_x}.
\]  

(B.104)

where

\[
G_h = \alpha \sigma (1 - \alpha)(1 + \varphi)((1 - 2\eta)(\eta\sigma - 1)\alpha + (\eta - 1)^2 + \eta\sigma(\gamma - 1) + \eta^2(\sigma - 1)),
\]  

(B.105)

\[
F_h = 4\alpha - \sigma - \varphi + 3\alpha\sigma + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - 6\alpha^2 + 44\alpha^2 - \alpha^4 - 3\alpha^2 + \alpha^2\sigma + 2\alpha\eta^2\sigma^2
\]  

(B.106)

\[
H_x = (\eta(1 - \alpha) + \gamma - 1)(1 - \alpha)^2 + \alpha\sigma[\eta(\eta(1 - \alpha) + 2(2\gamma - 1))(1 - \alpha) + \gamma(\gamma - 1)].
\]  

(B.107)
C Corollaries and Lemmas

C.1 Proof of Lemma 1

Log-linearization of equations (37), (36), (38), (25) around the deterministic steady state, where we utilize the independence of the terms of trade and technology shocks across time, gives

\[
\hat{Z}_t + \hat{N}_t = (1 - \alpha)(-\eta \hat{P}_{H,t} + \hat{C}_t + \eta \hat{P}_t) - \alpha \gamma \hat{P}_{H,t} \tag{C.1}
\]

\[
\hat{\mu}_t = \hat{Z}_t + \hat{P}_{H,t} - \sigma \hat{C}_t - \phi \hat{N}_t \tag{C.4}
\]

Thus, for complete markets, we obtain the following dynamics for \(\hat{Y}_t, \hat{C}_t, \hat{P}_{H,t}, \hat{P}_t, \hat{\mu}_t\):

\[
\hat{Y}_t = -\left[(1 - \alpha)\left(\frac{1 - \alpha}{\sigma} + \eta \alpha\right) + \alpha \gamma\right] \hat{P}_{H,t} \tag{C.5}
\]

\[
\hat{N}_t = -\left[(1 - \alpha)\left(\frac{1 - \alpha}{\sigma} + \eta \alpha\right) + \alpha \gamma\right] \hat{P}_{H,t} - \hat{Z}_t \tag{C.6}
\]

\[
\hat{C}_t = -\frac{1 - \alpha}{\sigma} \hat{P}_{H,t} \tag{C.7}
\]

\[
\hat{\mu}_t = (1 + \phi) \hat{Z}_t + (1 + \phi \gamma)(1 - \alpha)\left(\frac{1 - \alpha}{\sigma} + \eta \alpha\right) + \alpha \gamma) \hat{P}_{H,t} \tag{C.8}
\]

It is easy to see that \(\frac{\partial \hat{Y}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{N}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{C}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{\mu}_t}{\partial \hat{P}_{H,t}} > 0\), and with respect to technology shock we have \(\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} = 0\), \(\frac{\partial \hat{C}_t}{\partial \hat{Z}_t} = 0\), \(\frac{\partial \hat{N}_t}{\partial \hat{Z}_t} < 0\), \(\frac{\partial \hat{\mu}_t}{\partial \hat{Z}_t} > 0\). On the other hand, under financial autarky,

\[
\hat{Y}_t = [(1 - \alpha)(1 - \eta) - \gamma] \hat{P}_{H,t} \tag{C.9}
\]

\[
\hat{N}_t = [(1 - \alpha)(1 - \eta) - \gamma] \hat{P}_{H,t} - \hat{Z}_t \tag{C.10}
\]

\[
\hat{C}_t = -[(1 - \alpha)\eta + \gamma - 1] \hat{P}_{H,t} \tag{C.11}
\]

\[
\hat{\mu}_t = (1 + \phi) \hat{Z}_t + [\alpha(1 - \sigma) + (\sigma + \varphi)(-(1 - \alpha)(1 - \eta) + \gamma)] \hat{P}_{H,t} \tag{C.12}
\]

In this case, \(\frac{\partial \hat{Y}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{N}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{C}_t}{\partial \hat{P}_{H,t}} < 0\), \(\frac{\partial \hat{\mu}_t}{\partial \hat{P}_{H,t}} > 0\), and with respect to technology shock we have \(\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} = 0\), \(\frac{\partial \hat{C}_t}{\partial \hat{Z}_t} = 0\), \(\frac{\partial \hat{N}_t}{\partial \hat{Z}_t} < 0\), \(\frac{\partial \hat{\mu}_t}{\partial \hat{Z}_t} > 0\). ■
C.2 Proof of Lemma 2

Under flexible prices firms charge constant markups. Thus, for complete markets, we obtain the following dynamics for $\hat{Y}_t, \hat{C}_t, \hat{P}_{H,t}, \hat{P}_t$ by setting $\hat{\mu}_t = 0$ in the system (C.5)-(C.8):

\begin{align*}
\hat{Y}_t &= \frac{\delta_z + \varphi \delta_z \hat{Z}_t}{1 + \varphi \delta_z} \quad \text{(C.13)} \\
\hat{N}_t &= \frac{\delta_z - \frac{1}{1 + \varphi \delta_z} \hat{Z}_t}{1 + \varphi \delta_z} \quad \text{(C.14)} \\
\hat{C}_t &= \frac{1 - \alpha - 1 + \varphi}{\sigma - 1 + \varphi \delta_z} \hat{Z}_t \quad \text{(C.15)} \\
\hat{P}_{H,t} &= -\frac{1 + \varphi}{1 + \varphi \delta_z} \hat{Z}_t. \quad \text{(C.16)}
\end{align*}

where $\delta_z = (1 - \alpha)(\frac{1 - \sigma}{\sigma} + \eta \alpha) + \alpha \gamma > 0$.

It is easy to see that $\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} > 0, \frac{\partial \hat{C}_t}{\partial \hat{Z}_t} > 0, \frac{\partial \hat{P}_{H,t}}{\partial \hat{Z}_t} < 0$. On the other hand, under financial autarky,

\begin{align*}
\hat{Y}_t &= \frac{(1 + \varphi) \delta_{2,z}}{\alpha(1 - \sigma) + (\sigma + \varphi) \delta_{2,z}} \hat{Z}_t. \quad \text{(C.17)} \\
\hat{N}_t &= \frac{(1 - \sigma)(\delta_{2,z} - \alpha)}{\alpha(1 - \sigma) + (\sigma + \varphi) \delta_{2,z}} \hat{Z}_t, \quad \text{(C.18)} \\
\hat{C}_t &= \frac{(1 + \varphi)(\delta_{2,z} - \alpha)}{\alpha(1 - \sigma) + (\sigma + \varphi) \delta_{2,z}} \hat{Z}_t, \quad \text{(C.19)} \\
\hat{P}_{H,t} &= -\frac{1 + \varphi}{\alpha(1 - \sigma) + (\sigma + \varphi) \delta_{2,z}} \hat{Z}_t. \quad \text{(C.20)}
\end{align*}

where $\delta_{2,z} = (1 - \alpha)(\eta - 1) + \gamma$. In this case, $\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} > 0, \frac{\partial \hat{C}_t}{\partial \hat{Z}_t} > 0, \frac{\partial \hat{P}_{H,t}}{\partial \hat{Z}_t} < 0$.

C.3 Proof of Lemma 4

In the steady state, non-cooperative central banks maximize (34) subject to (36), (37), and (38), where $1_{CP} = 0$ and $1_{CM} = 0$ (asset market structure is irrelevant in the steady state). Thus, we can formulate a Lagrangian:

$$
\lambda_2((1 - \alpha) T^{-\eta} CP^n + \alpha T^{-\gamma} C_F - N) + \lambda_3((1 - \alpha) T^{1-\eta} + \alpha - P_t^{1-\eta}) + \lambda_4(N T - CP) \quad \text{(C.21)}
$$
The first order conditions for the problem with respect to consumption \(C_t\), labor \(N_t\), terms of trade \(\tilde{P}_{H,t}\), and real exchange rate \(\tilde{P}_t\) are given below:

\[
\frac{\partial L}{\partial C} = C^{-\sigma} + \lambda_2 (1 - \alpha) T^{-\eta} P^\eta - \lambda_4 P, \quad (C.22)
\]
\[
\frac{\partial L}{\partial N} = -N^\phi - \lambda_2 + \lambda_4 T, \quad (C.23)
\]
\[
\frac{\partial L}{\partial T} = -\eta (1 - \alpha) \lambda_2 T^{-\eta-1} C P^\eta - \gamma \alpha \lambda_2 T^{-\gamma} C_F + \lambda_3 (1 - \alpha) (1 - \eta) T^{-\eta} + \lambda_4 N, \quad (C.24)
\]
\[
\frac{\partial L}{\partial P} = (1 - \alpha) \eta \lambda_2 T^{-\eta} C P^\eta - (1 - \eta) \lambda_3 P^{-\eta} - \lambda_4 C = 0. \quad (C.25)
\]

Symmetric Steady State

In the symmetric steady state we use the fact that \(T = P = Z = 1\) and \(C = N\), then use these relationships to simplify first order conditions and obtain:

\[
\frac{\partial L}{\partial C_t} = C^{-\sigma} + \lambda_2 (1 - \alpha) - \lambda_4 = 0, \quad (C.26)
\]
\[
\frac{\partial L}{\partial N_t} = -C^\phi - \lambda_2 + \lambda_4 = 0, \quad (C.27)
\]
\[
\frac{\partial L}{\partial T_t} = -\eta (1 - \alpha) \lambda_2 C - \gamma \alpha \lambda_2 C + \lambda_3 (1 - \alpha) (1 - \eta) + \lambda_4 C, \quad (C.28)
\]
\[
\frac{\partial L}{\partial P_t} = (1 - \alpha) \eta \lambda_2 C - (1 - \eta) \lambda_3 C - \lambda_4 C. \quad (C.29)
\]

We simplify this system and use \(\frac{\partial L}{\partial P_t} (1 - \alpha) + \frac{\partial L}{\partial T_t} \):

\[
-\alpha(\eta(1 - \alpha) + \gamma) \lambda_2 + \alpha \lambda_4 = 0. \quad (C.30)
\]

We can express \(\lambda_4\) as function from \(\lambda_2\) from (C.30) and obtain \(\lambda_4 = (\eta(1 - \alpha) + \gamma) \lambda_2\). We substitute this result into the first order conditions (C.26)-(C.27) with respect to \(C\) and \(N\):

\[
C^{-\sigma} = ((\eta - 1) (1 - \alpha) + \gamma) \lambda_2, \quad (C.31)
\]
\[
C^\phi = (\eta (1 - \alpha) + \gamma - 1) \lambda_2. \quad (C.32)
\]
Expressing $\lambda_2$ in terms of $C, \gamma, \eta, \alpha$, and making these expressions equal allows to obtain:

$$C^{-\sigma} \left( \frac{1}{((\eta - 1)(1 - \alpha) + \gamma)} \right) = \frac{C^\phi}{(\eta(1 - \alpha) + \gamma - 1)}. \quad (C.33)$$

Thus, we can express $C$ in terms of $\alpha, \eta, \gamma$:

$$C^{-\sigma - \psi} = \frac{(\eta - 1)(1 - \alpha) + \gamma}{\eta(1 - \alpha) + \gamma - 1} = 1 + \frac{\alpha}{\eta(1 - \alpha) + \gamma - 1}. \quad (C.34)$$

As the relationship (25) in the steady state takes the form

$$C^{-\sigma - \psi} = \mu, \quad (C.35)$$

we can obtain the formula for the markup:

$$\mu = 1 + \frac{\alpha}{\eta(1 - \alpha) + \gamma - 1}. \quad (C.36)$$

**C.4 Proof of Corollary 4.2**

We begin with the expression for markup:

$$\hat{\mu}_t = \frac{G_x}{F_x H_x} \hat{E}_{s,t}. \quad (C.37)$$

where

$$G_x = \alpha \sigma (\sigma(1 - 2\alpha) + \gamma \sigma \eta - (1 - \alpha) \eta (\sigma + 2) + 1 - \alpha), \quad (C.38)$$

$$F_x = \alpha - 1 + \sigma (\gamma - \alpha \eta), \quad (C.39)$$

$$H_x = ((\eta(1 - \alpha) + \gamma - 1)(1 - \alpha)^2 + \alpha \sigma [\eta(\eta(1 - \alpha) + 2(2\gamma - 1))(1 - \alpha) + \gamma (\gamma - 1)]). \quad (C.40)$$

We start with the fact that for $\eta > 0, \gamma \geq 1, 0 < \alpha < 1$, we have $H_x > 0$. Second, using the solution from (B.99) and (B.103) we can express consumption dynamics in terms of export share and obtain:

$$\hat{E}_{s,t} = \frac{(1 - \alpha)(\alpha - 1 + \sigma (\gamma - \alpha \eta))}{(\alpha - 1)^2 + \alpha \sigma (\eta(1 - \alpha) + \gamma)} \hat{\gamma}_t. \quad (C.41)$$

The export share to be procyclical when $\alpha - 1 + \sigma (\gamma - \alpha \eta) > 0$, which is equivalent to $F_x > 0$. 

56
Let’s rearrange the expression $\frac{G_x}{F_x}$ and consider its simplified form:

$$\frac{G_x}{F_x} = \eta + \frac{(1 - \alpha)(\eta - \frac{1}{\sigma})(\eta - 1)}{(1 - \alpha)(\eta - \frac{1}{\sigma}) + (\gamma - \eta)} \quad (C.42)$$

As $F_x > 0$, the denominator in equation (C.42) is positive. If $\sigma > 1$, $\gamma \geq 1$, $0 < \alpha < 1$, we have to consider several cases for $\eta$. For example, if $\eta > \gamma$ and $F_x > 0$, then $(1 - \alpha)(\eta - \frac{1}{\sigma})(\eta - 1) > 0$ and $\frac{G_x}{F_x} > 0$. Second, if $1 < \eta < \gamma$, and then $\frac{G_x}{F_x}$ since the denominator and the numerator in equation (C.42) are positive. Third, it might be the case that $\frac{1}{\sigma} < \eta < 1$. In this case, the denominator in (C.42) is positive, while the numerator is negative. The whole expression $\frac{G_x}{F_x}$ monotonically increases with $\alpha$.

Let’s take a derivative with respect to $\alpha$:

$$\frac{\eta}{\gamma - \frac{1}{\sigma}} + \frac{1}{\sigma} (\eta - 1) > 0$$

Then let’s consider what happens at $\alpha = 0$, which sets $\frac{F_x}{G_x}$ to a minimum. In this case, we have

$$\frac{F_x}{G_x} = \eta + \frac{(\eta - \frac{1}{\sigma})(\eta - 1)}{\gamma - \frac{1}{\sigma}}. \quad (C.44)$$

We can multiply everything by the positive denominator and obtain the following expression:

$$\eta(\gamma - \frac{1}{\sigma}) + \eta^2 - (1 + \frac{1}{\sigma})\eta + \frac{1}{\sigma}, \quad (C.45)$$

which is equivalent to

$$(\eta - \frac{1}{\sigma})^2 + \eta(\gamma - 1) > 0 \quad (C.46)$$

Now we also technically have the case when $\eta < 1/\sigma$. In this case, as $F_x > 0$ and the numerator in $C.42$ is positive, we obtain $\frac{G_x}{F_x} > 0$. Thus, markups are procyclical if export share is procyclical for households with risk-aversion greater than one.
C.5 Proof of Corollary 4.3

We begin with the expression for markup:

\[ \hat{\mu}_t = \frac{G_x}{F_x H_x} \hat{E}_{s,t}, \]  

(C.47)

where

\[ G_x = \alpha \sigma (\sigma (1 - 2\alpha) \eta^2 + \gamma \sigma \eta - (1 - \alpha) \eta (\sigma + 2) + 1 - \alpha), \]  

(C.48)

\[ F_x = \alpha - 1 + \sigma (\gamma - \alpha \eta). \]  

(C.49)

\[ H_x = (\eta (1 - \alpha) + \gamma - 1)(1 - \alpha)^2 + \alpha \sigma \eta (\eta (1 - \alpha) + 2(2\gamma - 1))(1 - \alpha) + \gamma (\gamma - 1)]. \]  

(C.50)

As \( H_x > 0 \), we need to focus on \( G_x \) and \( F_x \). Under \( \eta = \gamma \), we can transform the expression to \( \hat{\mu}_t \) to

\[ \frac{\hat{\mu}_t}{\hat{E}_{s,t}} = \frac{\alpha \sigma (1 - \alpha) (2\gamma - 1) (\sigma \gamma - 1)}{(1 - \alpha) (\sigma \gamma - 1) H_x} = \frac{\alpha \sigma (2\gamma - 1)}{H_x} > 0. \]  

(C.51)

D Monetary Policy and Terms of Trade

The Euler equation for domestic home securities gives the following relationship:

\[ C_t^{-\sigma} = \beta R_t E_t \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}}. \]  

(D.1)

We can express inflation in the consumer price index in terms of producer price index in the following way:

\[ \pi_{t+1} = \frac{P_{F,t+1}}{P_{F,t}} \frac{\tilde{P}_{t+1}}{\tilde{P}_t} = \pi_{H,t+1} \frac{\tilde{P}_{t+1,1} \tilde{P}_{H,t}}{\tilde{P}_t \tilde{P}_{H,t+1}}. \]  

(D.2)

Note, that inflation \( \pi_{H,t+1} \) is predetermined at \( t \). Plugging this inflation in the original Euler equation gives

\[ \frac{\tilde{P}_{H,t+1} C_t^{-\sigma}}{P_t} = \beta \frac{R_t}{\pi_{H,t+1}} E_t \left[ C_{t+1}^{-\sigma} \frac{\tilde{P}_{H,t+1}}{\tilde{P}_{t+1}} \right]. \]  

(D.3)
Under complete markets we can rely on the expression for consumption

\[ C_{it} = \mathbb{E}\{ Y_{it} \tilde{P}_{H,it} \} \beta_{it}^{-\frac{1}{\sigma}}. \]  

(D.4)

and obtain:

\[ \tilde{P}_{H,t} = \beta \frac{R_t}{\pi_{H,t+1}} \mathbb{E}_t \tilde{P}_{H,t+1}. \]  

(D.5)

When the prices are set one period in advance, inflation dynamics on its own has no effect on the real economy. As a result, we are not interested in pinning down the inflation dynamics, using the dynamics of the real interest rate \( r_t = \frac{R_t}{\pi_{H,t+1}} \) for finding out the equilibrium of the real variables. Therefore, our objective is to generate a real interest rate rule that generates the path for the terms of trade \( \tilde{P}_{H,t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \mathbb{E}\tilde{P}_{H,t}. \) This is not a trivial exercise as we cannot rely on linear approximations due to the fact that we claim later in the paper to arrive at the optimality conditions non-linearly.

Moreover, our interest rate rule should rule out multiple equilibria. The following rule produces a unique stationary equilibrium that generates the necessary terms of trade dynamics:

\[ r_t = 1 - \frac{\delta}{\beta} \mathbb{E}_t \left[ \mathbb{E}\left[ \tilde{P}_{H,t} \left( \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \right)^{1+\delta} \right] \right]. \]  

(D.6)

where \( \delta \) is a positive constant. We use this equation as more simple rule \( r_t = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \) cannot rule out sunspot terms for the terms of trade. This rule for the interest rate allows to obtain the modified relationship for the terms of trade:

\[ \tilde{P}_{H,t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \mathbb{E}\tilde{P}_{H,t}. \]  

(D.7)

For this equation, the only non-explosive solution for the terms of trade dynamics is \( \tilde{P}_{H,t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \mathbb{E}\tilde{P}_{H,t}. \)

Under financial autarky, we return to the original Euler equation and set the interest rate rule as \( R_t = \pi_{t+1} + \beta g(Z_t) \mathbb{E}(g(Z_t)) \) and using the independence of shocks across time generates:

\[ C_t - \sigma \frac{\tilde{P}_{H,t}}{P_t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} \mathbb{E}\left[ C_t - \sigma \frac{\tilde{P}_{H,t}}{P_t} \right]. \]  

(D.8)
Having the control over the cyclical dynamics \( C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t} \), coupled with the constraints (36)-(38) allows us to extract the dynamics for the terms of trade. For example, we can express consumption under financial autarky using (36)-(38) as

\[
C_t = ((1 - \alpha) \tilde{P}_{H,t}^{1-\eta} + \alpha) \frac{\tilde{P}_{H,t}^{1-\gamma} A}{1-\gamma} \tag{D.9}
\]

where \( A \) is a constant. Using this relationship, we can obtain the expression for \( C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t} \):

\[
C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t} = ((1 - \alpha) \tilde{P}_{H,t}^{1-\eta} + \alpha) \frac{\tilde{P}_{H,t}^{\gamma\sigma - \sigma + 1}}{1-\gamma} A^{-\sigma} \tag{D.10}
\]

Thus, the desire to replicate a particular cyclical dynamics of the terms of trade \( \frac{\tilde{P}_{H,t}}{P_t} = f(Z_t) \) requires to generate the dynamics for

\[
\frac{C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t}}{E[C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t}]} = \frac{((1 - \alpha) f(Z_t)^{1-\eta} + \alpha E f(Z_t))^{\gamma\sigma - \sigma + 1}}{E[((1 - \alpha) f(Z_t)^{1-\eta} + \alpha E f(Z_t))^{\gamma\sigma - \sigma + 1}]} \tag{D.11}
\]

Defining \( g(Z_t) \) as \( ((1 - \alpha) f(Z_t)^{1-\eta} + \alpha E f(Z_t))^{\gamma\sigma - \sigma + 1} \), we can use a similar rule

\[
r_t = \frac{1}{\bar{\beta} E[(C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{P_t})^{-\delta} (\frac{g(Z_t)}{E g(Z_t)})^{1+\delta}]} \tag{D.12}
\]

Consequently, the central bank can control the terms of trade over the cycle under financial autarky, and when markets are complete.