How Misleading is Linearization?

Evaluating the Dynamics of the Neoclassical Growth Model*

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Abstract

The standard procedure for analyzing transitional dynamics in non-linear macro models has been to employ linear approximations. This raises the central question of this paper: How reliable is this procedure in evaluating the dynamic adjustments to policy changes or structural shocks? This question is significant since one of the basic objectives of contemporary micro-based macroeconomic models is the analysis of intertemporal welfare. We analyze this issue in the context of a neoclassical Ramsey growth model, with two alternative specifications of productive government spending, by employing both linearization and non-linear solution techniques. We find that if government expenditure is introduced as a flow and the dynamic adjustment is fast, linearization may be a reasonably good approximation of the true dynamics even for fairly large policy shocks. In contrast, if government expenditure assumes the form of a stock, leading to more sluggish adjustment, linearization is more problematic. The linearization procedure may yield misleading predictions, both qualitatively and quantitatively. These occur at the beginning of the transition and therefore weigh heavily in intertemporal welfare calculations. These patterns are verified for temporary shocks as well.

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1. Introduction

The study of macroeconomics has been revolutionized over the past forty years or so. For years the static IS-LM framework provided the basic paradigm, although early work by Ott and Ott (1965), Christ (1967, 1968) and others, sought to embed this approach in a dynamic framework linking the accumulation of stocks and flows.1 Starting around 1980, the arbitrariness of these early models began to be seriously questioned. Critics argued that a good macro model should be based on sound microeconomic foundations, which typically involved deriving the equilibrium structure of the economy from the intertemporal optimization of the micro-agents comprising the economy. This has led to the so-called representative agent model, which is arguably the standard workhorse macro model of today; see e.g. Blanchard and Fischer (1989) and Turnovsky (2000) for textbook treatments. However, this approach is not without its critics and indeed the effort to move beyond this paradigm is currently an area of intensive research activity.2

One characteristic of the macroeconomic equilibrium derived from intertemporal optimization is that it is inevitably nonlinear. Diminishing marginal utility on the consumption side and diminishing marginal physical productivity on the production side guarantees that this will be the case. While this does not create much of a problem for the characterization of the static short-run or steady-state equilibrium, for which well developed analytical and numerical algorithms exist, it does pose severe difficulties for the analysis of the transitional dynamics of the system. If one formulates macroeconomic models using continuous time, this involves the analysis of nonlinear differential equation systems, which in general do not have closed-form solutions. Models that are specified in discrete time and employ difference equations may be subject to even more acute technical problems.3

The standard procedure for analyzing the transitional dynamics following a structural change or policy shock is to linearize the dynamic system around its (post-shock) steady state and then to

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1 Building on these earlier contributions, Turnovsky (1977) called this the “intrinsic dynamics” of the macroeconomic system, contrasting that with the dynamics arising from arbitrarily specified lags, such as sluggish adjustments.
2 See e.g. Kirman (1992), Hommes (2006), and in particular the papers in Colander (2006).
3 Gray and Turnovsky (1979) argued that if one derives the macroeconomic equilibrium as the aggregation over a continuum of agents, each of whom transacts over a discrete unit of time (say once a week), aggregate behavior is likely to be represented by a mixed differential-difference equation, which is even more intractable.
study the dynamic properties of this linearized system as an *approximation* to the true non-linear system. This raises the central question of this paper: How reliable (or unreliable) is the linear approximation, especially when applied to non-linear growth models for the purposes of evaluating the dynamic response of an economy to policy changes or structural shocks? While we know that linearization involves errors, how serious are they? These questions assume critical significance when one considers that one of the basic objectives of micro-based macrodynamic models is to analyze the consequences of policy for intertemporal welfare.

Clearly, as long as the structural or policy changes are small, the linear approximation should be adequate and give reasonably accurate guidance to the “true” dynamic response of the economy. Thus, if we are dealing with transitory shocks and short-run dynamics, when the economy is likely to remain quite close to its steady state, the linear approximation should provide a reasonable approximation to the true dynamic response.

But one can easily envision scenarios where this need not be the case. For example, consider the following two situations. In the first, the economy undergoes substantial policy reforms that are associated with large structural adjustments. In the second, the dynamic evolution of the economy is characterized by multiple state variables (e.g. physical and human capital, infrastructure, and debt), the adjustments of which compound to render an overall sluggish dynamic process. In both of these examples the economy may remain far away from its long-run equilibrium for extended periods of time, and therefore to analyze the dynamic response of the economy by linearizing around the steady-state equilibrium may potentially be quite misleading. Yet this is the dominant practice adopted in the macro-growth literature, and has been widely applied in a range of areas, where the concern has been to characterize the specific transitional path to the steady state.\(^4\)

Whether explicit computation of the equilibrium transitional path is necessary depends upon the intended use of the model. For example, the conventional restrictions on the utility and

\(^4\) Modern macro-dynamic models, based on intertemporal optimization, are typically characterized by “saddle-point” behavior, and to establish the extent to which this prevails involves identifying and analyzing the configuration of the eigenvalues of the associated linear system. Examples of this, spanning a variety of areas of application, include Blanchard and Fischer (1989), Azariadis (1993), Sargent (1979), Turnovsky (2000), Baxter and King (1993), Futamagi, et al (1993), Bond, Wang, and Yip (1996), Agenor and Montiel (1996), Otigueira and Santos (1997). The earlier literature, which focused on “rational expectations”, is largely restricted to linear systems; see Sargent (1979), Turnovsky (2000, Part II).
production functions suffice to provide a solid understanding of the global qualitative properties of
the transitional behavior of the standard one-sector Ramsey model; see Blanchard and Fischer
(1989). On the other hand, the explicit computation of the transitional path becomes necessary if
one wishes to quantify the impacts of specific policy and other structural changes on the structure
and performance of the economy, as it evolves through time. Furthermore, numerical simulations of
the transitional dynamics also become inevitable as the specification of the macro-dynamic system is
enriched, and its dimensionality increases.5

At the same time, while research on macroeconomic dynamics has been evolving in the
direction suggested, computational methods have been developing as well, with the numerical
solution techniques necessary to solve differential equation systems improving significantly in recent
years. Moreover, the dramatic increase in computing capacity over the last several years has
significantly enhanced our ability to solve numerically nonlinear differential equation systems. Most
of these have involved some form of “shooting algorithm” and projection methods; see Judd (1998)
for an extensive discussion. However, as Judd notes, saddle-point equilibria typically associated
with intertemporal optimizing behavior are difficult to compute numerically. Forward-shooting
algorithms, which involve a guess for the initial value of the jump variable, may lead to substantial
errors in long-run steady states, especially where multiple jump variables are present. Reverse-
shooting algorithms, recommended by Judd (1998) and Brunner and Strulik (2002), while attractive,
have been difficult to apply due to the limited progress in developing algorithms for systems
containing multiple state variables. Thus, despite developments in computational techniques,
linearization remains the prevailing procedure for analyzing dynamic time paths, especially in the
macroeconomics literature.

Recently, Atolia and Buffie (2007) have sought to make the reverse-shooting algorithm more
applicable for dynamic models containing up to four state variables. They do this by refining the
search algorithm, which uses sufficient information to always converge to the true solution. More
importantly, they seek to promote the use of reverse shooting methods by developing canned
programs that can be readily implemented using Mathematica or some other program, thus making

5 The use of calibration methods is of course central to the real business cycle methodology; see e.g. Cooley (1995).
them user-friendly in a way that canned econometric packages are. Being able to handle up to four state variables efficiently is important for economic growth theory, since one can immediately conceive of private capital, public capital, human capital (knowledge), government debt as four natural state variables, all of which interact in the growth process.

An interesting (and practical) question that arises from all this is how misleading is the process of linearization of the dynamic path? Does it lead to implications that are not only quantitatively, but also qualitatively, erroneous? The purpose of this paper, therefore, is to compare the numerical predictions of a linearized dynamic model with those implied by a non-linear solution technique. In this respect it is related to earlier work by Karakitsos and Rustem (1989), who characterized the unreliability of linear approximations relative to non-linear algorithms for versions of the IS-LM model. More recently, Becker, Grune, and Semmler (2007) compare the performance of dynamic programming and second-order approximation methods for dynamic general equilibrium models by employing a model for which an exact analytical solution is available. However, our approach differs from Karakitsos and Rustem (1989) in that we consider models based on intertemporal optimization. Moreover, by employing a more general model for which no closed-form analytical solution is available, it also contrasts with Becker, Grune, and Semmler (2007). Instead, our comparison is with a very accurate numerical solution obtained by employing the reverse-shooting technique. We restrict attention to an evaluation of the performance of linearization, as it is the dominant method in literature on growth and development.6

Since the difference between linear and non-linear solution techniques will depend critically on the intrinsic structure of a model (such as the interaction of state variables, presence of other sources of sluggishness such as adjustment costs or capital market imperfections, etc.), it is difficult to derive general conclusions about the implications of linearization. Thus, while our approach is illustrative, nevertheless we are able to identify some very robust patterns arising from linearization in the context of our model.

The model we employ is a variant of the basic one-sector neoclassical Ramsey growth

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6We should also note the work of Kim and Kim (2003) that compares first order approximations to second order approximations.
model, augmented to allow for productive government spending.\(^7\) We consider two widely adopted versions of the model. In the first, government spending is introduced as a flow in production, as is the case in much of the recent endogenous growth literature, spawned by Barro’s (1990) seminal contribution. In this case, the macro-dynamic equilibrium is generated by a one dimensional nonlinear differential equation in the private capital stock. However, this specification has been subject to the criticism that since productive government spending falls largely on the economy’s infrastructure, it is more appropriately represented as a stock rather than as a flow. Hence, the second formulation of the model introduces public capital as a stock, along with private capital, so that the macroeconomic equilibrium now involves two state variables.\(^8\)

In both cases, we first calibrate the model and then solve it numerically using both linear approximation and non-linear techniques. The non-linear solution is obtained by applying the Atolia and Buffie (2007) form of the reverse shooting algorithm, while the linearized solution involves a standard first-order Taylor series approximation. We introduce an increase in the rate of productive government spending. In order to test the sensitivity to linearization, we consider two alternative increases in the rate of public investment: (i) from 4% to 5%, (as a share of output) and (ii) from 4% to 8%. The first case, which we consider to be a “moderate” increase, represents a 25% increase in the policy variable (the current level of government expenditure), while the second is a doubling of the rate of public investment and we characterize this as a “large” increase. We then compare the dynamic time paths of the key variables of the model and the welfare changes implied by the linear and non-linear approximation solution methods of the transitional paths. Since the non-linear dynamic paths are the closest approximation to the “true” or global dynamic path, any deviation of the “linearized” dynamic paths from the non-linear paths is characterized as an “error.”\(^9\)

As expected, we find that the farther an economy is from its steady-state equilibrium, the larger are the errors generated by linearization, both qualitatively and quantitatively. The time and

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\(^7\) The Ramsey growth model with government spending can be traced back to Arrow and Kurz (1970) and has since become a standard work-horse model for analyzing the effects of fiscal policy on macroeconomic performance, with wide applications in both the business cycle and growth literature.


\(^9\) We have also computed the dynamic paths using alternative projection methods and the results are the same as our shooting algorithm to the 5\(^{th}\) decimal place. We are therefore confident that our nonlinear dynamic paths are very close to the true optimal path.
distance from steady state (and therefore the associated errors from linearization) are closely related to the speed of convergence, and in this regard we find that the two policy specifications yield significant differences. When government spending is introduced as a flow, the transition is shorter, and the economy converges more rapidly to its steady-state equilibrium (due to the one-dimensional transition path). The errors from linearization are relatively small and decline quite rapidly as the economy approaches its new steady state.

In contrast, when government expenditure is introduced in the form of public capital (stock), the errors from linearization are more substantial. The linearized model consistently over-predicts consumption and welfare gains from an increase in public investment in infrastructure. Errors are much larger and are concentrated in the initial phase of the dynamic adjustment. Because the introduction of public capital as a second state variable significantly reduces the speed of convergence, this can compound to substantial long-run errors in computing welfare. Furthermore, in many cases, we find that the errors may even be qualitative: for example, when the “true” (non-linear) solution implies a short-run decline in consumption, the linearized approximation predicts an increase. If this occurs, the use of linearization methods may yield quite misleading predictions for quantities such as intertemporal welfare, which place heavy weight on the instantaneous responses of consumption and labor supply.

We also examine the consequences of linearization for the speed of convergence, a subject of extensive theoretical and empirical investigation in recent years. We find that linearized models not only tend to over-predict the speed of convergence for substantial portions of the transition path, but also provide qualitatively contrasting time profiles for the speed of adjustment relative to non-linear models. Indeed, in a recent paper, Papageorgiou and Perez-Sebastian (2007) point out that the asymptotic speed of convergence implied by a linearized two-sector growth model is not able to reproduce actual convergence experiences of countries such as Japan and South Korea. Therefore, their results lend support to our finding that the speed of convergence implied by linearization can be

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10 Empirical estimates of the speed of convergence have typically been much lower than those implied by theory, where a frequently used measure is the stable eigenvalue of a linearized dynamic system; See Mankiw et al. (1992), Barro and Sala-i-Martin (1992) for early empirical work on this issue. Theoretical studies by Ortiguera and Santos (1997), Eicher and Turnovsky (1999), and Chatterjee (2005) introduce various modifications into standard growth models that reduce the speed of convergence to those implied by empirical studies.
a misleading indicator of the *true* speed of convergence of a dynamic system.

Finally, we compare the predictions of the linearized and non-linear solutions when government spending shocks are temporary in nature, thus focusing on the effect of the *duration* of a shock for the underlying solution technique. We find that although the errors from linearizing the flow model remain tolerable for the Cobb-Douglas case, the magnitudes of the errors increase for higher values of the elasticity of substitution in production.\textsuperscript{11} For the stock model, the errors from linearization are more magnified compared to those generated by permanent shocks.

The remainder of the paper is as follows. In Sections 2 and 3 we lay out the theoretical models underlying our analysis. Section 4 discusses the solution of the global dynamics and Section 5 presents the numerical comparisons between linearized and non-linear solutions to the theoretical models. Section 6 presents similar comparisons for temporary shocks, and finally, Section 7 concludes, and briefly addresses some consequences of linearization for other important issues in macrodynamics.

### 2. The One-State Variable Case: Public Investment as a “Flow”

We begin with the simpler and more common specification, where productive government expenditure is introduced as a flow in the production function, as in Barro (1990) and much of the subsequent endogenous growth literature.

#### 2.1 Analytical Framework

Consider an infinitely-lived representative agent, modeled as a consumer-producer, who maximizes utility from consumption (*C*) and leisure (*I*):

\[
\text{Maximize } \int_0^\infty \frac{1}{\gamma} (C^\theta)^\gamma e^{-\beta t} dt, \quad -\infty < \gamma < 1, \theta > 0
\]

The agent’s optimization is subject to the following budget constraint:

\[11\] This may seem counterintuitive. It implies that as the elasticity of substitution increases and the production function becomes more linear, linearization in response to the policy shock performs more poorly. The reason is that the higher the elasticity of substitution, the larger the change in the steady state so that the farther the economy is initially from its new steady state.
\[ \dot{K} = (1 - \tau)Y - C - T - \delta K \]  

(2)

where \( K \) denotes the private capital stock that depreciates at the constant rate \( \delta \), \( \tau \) is the rate of income tax, \( T \) is lump-sum taxation, and \( Y \) is the flow of output produced by the agent using a Constant Elasticity of Substitution (CES) production technology:\(^{12}\)

\[ Y \equiv Y(K, l) = A[\alpha\{G'(1 - l)\}^{-\rho} + (1 - \alpha)K^{-\rho}]^{\frac{1}{\rho}}, \quad A > 0, \quad 0 < \alpha < 1, \quad 0 < \eta < 1 \]  

(3)

The production function is homogeneous of degree one with respect to private capital, \( K \), and labor supply, \((1 - l)\). Productive government expenditure, \( G \), is introduced as a flow that interacts directly with labor to enhance its productivity.

The flow of public expenditures is specified as a fixed fraction of output

\[ G = gY, \quad 0 < g < 1 \]  

(4a)

with an increase in \( g \) specifying an expansion in expenditure. This specification is natural in a growth context. The government finances its expenditure using a combination of a proportional income tax and lump-sum taxation, while maintaining a balanced budget at all points of time:

\[ G = \tau Y + T \]  

(4b)

Combining (2), (4a), and (4b), yields the goods market equilibrium condition

\[ \dot{K} = (1 - g)Y - C - \delta K \]  

(4c)

### 2.2 Consumer Optimization

Performing the optimization yields the following first order conditions:

\[ C^{r - 1}t^\rho = \hat{\lambda} \]  

(5a)

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\(^{12}\) This formulation serves as an adequate vehicle to illustrate the computational issues we wish to discuss. The introduction of government spending as labor-augmenting has the well-known convenience of ensuring that the economy has a balanced growth path if the model is extended to introduce population growth or technological change. For the Cobb-Douglas production function, which serves as our benchmark, how government spending is introduced is unimportant.
\begin{align}
\theta C^\rho l^{\theta-1} &= -\lambda(1-\tau) \frac{\partial Y}{\partial l} = \lambda(1-\tau)\alpha(Ag^\eta)^{-\rho} \frac{Y^{1+\rho(1-\eta)}}{(1-\lambda)^{1+\rho}} \tag{5b} \\
\frac{\beta}{\lambda} \frac{\dot{\lambda}}{\lambda} &= (1-\tau) \frac{\partial Y}{\partial K} - \delta_k = (1-\tau)(1-\alpha)A^{-\rho} \left(\frac{Y}{K}\right)^{1+\rho} - \delta_k \tag{5c}
\end{align}

together with the transversality condition
\[
\lim_{t \to \infty} \lambda Ke^{-\beta t} = 0 \tag{5d}
\]

These conditions are routine. Equation (5a) equates the marginal utility of consumption to the shadow value of capital, (5b) equates the marginal utility of leisure to the income foregone measured in utility units, while (5c) equates the marginal utility of consumption to the after-tax rate of return on capital.

2.3 Equilibrium Dynamics

The equilibrium dynamics can be described in terms of the evolution of the stock of private capital, $K$, and leisure, $l$.\textsuperscript{13} To derive the dynamics of capital accumulation is very straightforward. We first divide (5b) by (5a) to get the marginal rate of substitution between consumption and leisure:
\[
\frac{C}{l} = \frac{\alpha(Ag^\eta)^{-\rho} (1-\tau) Y^{1+\rho(1-\eta)}}{\theta (1-\lambda)^{1+\rho}} \tag{6}
\]
Combining this expression with the production function enables us to express $C$ in the form
\[
C \equiv C(l, K; g, \tau)
\]
so that substituting into (4c) yields
\[
\dot{K} = (1-g)Y(K, l) - C(l, K; g, \tau) - \delta_k K \tag{7a}
\]
The derivation of the dynamics of $l$ is more involved. First, we first differentiate (5a) with respect to time, and combine with (5c), to obtain:

\textsuperscript{13} The structure is virtually identical to that of Turnovsky (2002), except that the economy has been augmented by the addition of a government fiscal sector.
\[
(\gamma - 1) \frac{\dot{C}}{C} + \theta \frac{\dot{i}}{l} = \beta + \delta_K - (1 - \tau)(1 - \alpha)A^{-\rho} \left( \frac{Y}{K} \right)^{1+\rho}
\]  

(8)

Next, differentiating (6) with respect to time gives

\[
\frac{\dot{C}}{C} = \frac{\dot{i}}{l} = (1 + \rho) \left( \frac{l}{1 - l} \right) \frac{\dot{i}}{l} + [1 + \rho(1 - \eta)] \frac{\dot{Y}}{Y}
\]

(9)

and taking the time derivative of the production function yields

\[
\frac{\dot{Y}}{Y} = A^{-\rho} \left[ \frac{g^{\rho}(1 - l)\rho}{g^{\rho}(1 - l)^{\rho} - A^{-\rho} \alpha \eta Y^{\rho(1 - \eta)}} \right] \left[ \frac{\alpha Y^{\rho(1 - \eta)}}{g^{\rho}(1 - l)^{\rho} - \alpha \eta A^{-\rho} Y^{\rho(1 - \eta)}} \right] + (1 - \gamma)(1 - \alpha) \{1 + \rho(1 - \eta)\} \left[ \frac{A^{-\rho} Y^{\rho(1 - \eta)}}{g^{\rho}(1 - l)^{\rho} - \alpha \eta A^{-\rho} Y^{\rho(1 - \eta)}} \right]
\]

(10)

Eliminating \( \dot{C}/C \) and \( \dot{Y}/Y \) from (8), (9), and (10), we can express the equilibrium dynamics for leisure as:

\[
\dot{i} = \frac{J(l, K, \dot{K})}{H(l, K)}
\]

(11)

where,

\[
J(l, K, \dot{K}) = \left[ \beta + \delta_K - (1 - \tau)(1 - \alpha)A^{-\rho} \left( \frac{Y}{K} \right)^{1+\rho} + (1 - \gamma)(1 - \alpha) \{1 + \rho(1 - \eta)\} \left( \frac{Y}{K} \right)^{\rho} \left( \frac{\dot{K}}{K} \right) \right]
\]

(12a)

\[
H(l, K) = \left[ \gamma(1 + \theta) - 1 \right] - (1 - \gamma) \left[ (1 + \rho) \left( \frac{l}{1 - l} \right) + [1 + \rho(1 - \eta)] \left\{ \frac{A^{-\rho} Y^{\rho(1 - \eta)}}{g^{\rho}(1 - l)^{\rho} - \alpha \eta A^{-\rho} Y^{\rho(1 - \eta)}} \right\} \right]
\]

(12b)

Finally, using (7a) to substitute for \( \dot{K} \) in (11) yields an equation for \( \dot{i} \) of the form

\[
\dot{i} = \frac{A(l, K)}{B(l, K)}
\]

(7b)

The pair of equations (7a) and (7b) are a highly nonlinear pair of autonomous differential equations in \( K \) and \( l \). This is just the conventional Ramsey model, modified by endogenous labor supply, and augmented with productive government spending. The standard procedure is to linearize these equations about the steady-state given in (13) below, when it can be shown to be a saddlepoint, with a one-dimensional stable manifold. Our task ahead is to assess the inaccuracies associated with this
approximation.

2.4. Steady-State Equilibrium

The steady-state equilibrium is attained when \( \dot{K} = \dot{l} = 0 \). The corresponding conditions are

\begin{align}
(1 - g)\bar{Y} &= C(\bar{l}, \bar{K}) + \delta_k \bar{K} \\
(1 - \tau)(1 - \alpha)A^{-\rho} &\left(\frac{\bar{Y}}{\bar{K}}\right)^{1+\rho} = \beta + \delta_k
\end{align}

Equations (13a) and (13b) can be solved for the steady-state values \( \bar{K} \) and \( \bar{l} \). Once these are known, the steady-state levels of consumption and output can immediately be determined from (6) and the production function.

3. The Two-State Variable Case: Public Investment as a “Stock”

While introducing productive government expenditure as a flow is analytically convenient in the sense of limiting the dimension of the model, it may be an inadequate representation of public infrastructure, which is more appropriately represented as a capital stock. To incorporate this we modify the production function (3) to

\begin{equation}
Y = A[\alphaIZES{(1 - l)}^{-\rho} + (1 - \alpha)K^{-\rho}]^{1/\rho}, A > 0, 0 < \alpha < 1, 0 < \eta < 1
\end{equation}

where \( K_G \) represents the aggregate stock of public capital. The agent’s optimization problem remains the maximization of (1) subject to (2), where the production function is now replaced by (14). The accumulation of public capital takes place according to

\begin{equation}
\dot{K}_G = G - \delta_G K_G = gY - \delta_G K_G, 0 < \delta_G < 1
\end{equation}

where \( \delta_G \) is the rate of depreciation.

The main change introduced by this re-specification of government expenditure is that the
macrodyanmic equilibrium is represented by a third-order dynamic system involving the evolution of private capital, \( K \), public capital, \( K_G \), and leisure \( l \). The corresponding dynamic equations are

\[
\dot{K} = (1 - g)Y - C(l, K, K_G) - \delta_K K \quad (16a)
\]

\[
\dot{K}_G = gY - \delta_G K_G \quad (16b)
\]

\[
\dot{l} = \frac{J(l, K, K_G, \dot{K}, \dot{K}_G)}{M(l, K, K_G)} \quad (16c)
\]

where,

\[
C \equiv C(l, K, K_G) = \frac{\alpha A^{-\rho}(1 - \tau)lK^{1-\rho}}{\theta Y \left(1 - l\right)^{1+\rho}}
\]

\[
J(l, K, K_G) = \left[ \beta + \delta_K - (1 - \tau)(1 - \alpha)A^{-\rho}\left(\frac{Y}{K}\right)^{1+\rho} + (1 - \gamma)(1 + \rho)\left(\Delta_1 \frac{\dot{K}}{K} + \Delta_2 \frac{\dot{K}_G}{K_G}\right) \right]
\]

\[
M(l, K, K_G) = \theta \gamma - (1 - \gamma) \left[ 1 + (1 + \rho) \left( \frac{l}{1 - l} + \alpha A^{-\rho} \left(\frac{Y}{K_G} \left(1 - l\right)\right)^\rho \right) \right]
\]

\[
\Delta_1 = (1 - \alpha) A^{-\rho} \left(\frac{Y}{K}\right)^\rho, \quad \Delta_2 = \eta \left[ \alpha A^{-\rho} \left(\frac{Y}{K_G} \left(1 - l\right)\right)^\rho - \frac{\rho}{1 + \rho} \right]
\]

This system is a slight generalization of that analyzed by Turnovsky (2004).\(^\text{14}\) Formal analysis of the linearized approximation proves to be intractable, although Turnovsky found for extensive simulations that it yielded two negative and one positive eigenvalue, implying a unique stable adjustment path. In this case, the stable manifold, being two-dimensional, allows for much more flexible dynamic behavior and can be reconciled more broadly with empirical evidence.\(^\text{15}\)

The steady-state equilibrium is attained when \( \dot{K} = \dot{K}_G = \dot{l} = 0 \). The corresponding conditions are

\[
(1 - g)\bar{Y} = C(\bar{l}, \bar{K}, \bar{K}_G) + \delta_K \bar{K} \quad (17a)
\]

\(^\text{14}\) Turnovsky (2004) restricted the production function to be of the Cobb-Douglas form.
\(^\text{15}\) This issue is emphasized by Eicher and Turnovsky (1999) in another context.
\[ g\tilde{Y} = \delta_g K_g \]  \hspace{1cm} (17b)

\[ (1-\tau)(1-\alpha)A^{\tau} \left( \frac{\tilde{Y} - Y}{K} \right)^{1+\rho} = \beta + \delta_k \]  \hspace{1cm} (17c)

where

\[ \tilde{Y} = A \left[ \alpha \left( \tilde{K}_G^\rho (1-I) \right)^{-\rho} + (1-\alpha)\tilde{K}^{-\rho} \right]^{1/\rho} \]

\[ \frac{\tilde{Y}}{K} = A \left[ (1-\alpha) + \alpha \left( \frac{\tilde{K}_G^\rho (1-I)}{K} \right)^{-\rho} \right]^{-\frac{1}{\rho}} \]

Equations (17a)-(17c) can in principle be solved for \( \tilde{K} \), \( \tilde{K}_G \), and \( \tilde{I} \).

4. Solving for the Global Nonlinear Saddle Path

As we have noted, in the existing literature on growth and development, the transitional dynamics of a model such as outlined in previous sections is almost without exception analyzed by linearizing the system around its steady-state equilibrium. This procedure is likely acceptable if the economy is subject to only small stationary shocks and therefore remains “close” to the steady state. This is likely to be true for the analysis of business cycles. However, applications pertaining to growth and development are more likely to consider issues such as policy reform leading to substantial structural changes; see Rodrik (1996). In this case linearization may cease to be reliable, and the need for a technique to evaluate the global non-linear transition paths arises.

As discussed in the introduction, there are several methods that can be employed to study non-linear dynamics, though they have received surprisingly little or no attention in the field of growth and development. Atolia and Buffie (2007) attempt to bridge the gap between existing numerical methods and their applications in macroeconomics. Their objective has been to develop reliable and user-friendly programs that can be used by an average practitioner with experience in higher-level programming environments such as Mathematica, Matlab, etc.\(^\text{16}\)

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\(^\text{16}\) These programs are available in the public domain at http://mailer.fsu.edu/~matolia/. Additional programs solving models with 2-3 jump variables, and a variety of shocks categorized by whether they are temporary or permanent as also available. Most common models in growth and development fall into at least one of the categories covered by these.
The programs are based on the underlying principle of “shooting” and can be used to solve a wide-variety of problems containing 2-4 state variables.\textsuperscript{17} Also, with the availability of cheap computing power, computation times are quite reasonable. For lower-order systems, computing times are of the order of a few minutes to a few tens of minutes. For higher-order systems, it may stretch to several hours. We employ one of the Atolia-Buffie program (\textit{Reverse Shoot 2D}) to solve the models of this paper. We should emphasize that the idea of (reverse) shooting is not new.\textsuperscript{18} What is new is the algorithm used to implement shooting that makes this numerical technique more accessible to the profession.

5. Numerical Analysis

In evaluating the accuracy of linearization, a basic issue that we need to address concerns the assessment of the errors. In doing so, our objective is to present them in the most informative way. The issue is the following. Our focus is on analyzing changes in the economy and evaluating the errors committed when they are computed as a linear approximation along a transitional path. A seemingly natural measure of the error associated with a variable, $x$ say, is the quantity,

$$\frac{[x(T) - x(T)]}{[x(T) - x_0]}$$

where $x(T)$ is value of $x$ at time $T$ predicted by the linearized model, $x_0$ is the initial value of $x$, and $x(T)$ is the actual (true) value obtained from solving the nonlinear model. The problem is that this measure, which is expressed as a percentage of the true change, can misleadingly suggest huge errors, when in fact they are quite small. This is particularly true for the higher-order model which may generate non-monotonic behavior during the transition. Thus, if starting from $x_0$, the true path for $x$ crosses its initial value, $x_0$ during the transition, at some point $T_1$, then around $t = T_1$, (18) will imply huge errors (infinite at $t = T_1$), even if $x(t)$ is tracking $x(t)$ closely. For this reason, we

\textsuperscript{17} Judd (1998) is a classic reference for numerical methods relevant for economic models. Also see Brunner and Strulik (2002) for endogenizing the duration of backward integration in shooting methods.

\textsuperscript{18} Early applications include Lipton, et al. (1982).
prefer to measure the errors associated with linearization by
\[
\frac{[x_i(T) - x(T) - x]}{x_0},
\]
which is expressed as a percentage of the initial pre-shock equilibrium value. In any event, the graphical illustrations for the time paths give a good intuitive sense as to the accuracy of the linearization procedure and where it is most vulnerable.

5.1. The “Flow” Model: One State Variable Case

We begin by reporting the numerical results obtained for the flow version of the model set out in Section 2. These are based on the following base parameters:

<table>
<thead>
<tr>
<th>Table 1: Base Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taste parameters: ( \theta = 1.75, \beta = 0.04 )</td>
</tr>
<tr>
<td>Technological parameters: ( \alpha = 0.667, A = 1, \eta = 0.20, \delta_k = 0.05 )</td>
</tr>
<tr>
<td>Policy parameters: ( \tau = 0.20, g_0 = 0.04 )</td>
</tr>
</tbody>
</table>

These parameters are all non-controversial. The rate of time preference, \( \beta = 0.04 \), distributive share of labor, \( \alpha = 2/3 \), the elasticity of leisure in utility, \( \theta = 1.75 \), rate of depreciation, \( \delta_k = 0.05 \), are all standard.\(^{19}\) The average tax rate, \( \tau = 0.20 \), and rate of government investment expenditure of \( g_0 = 0.04 \), is consistent with empirical evidence, while the productivity elasticity of government spending \( \eta = 0.20 \), is also conventional and plausible.\(^{20}\) Limited sensitivity analysis is conducted by varying the elasticity of substitution, \( \sigma \), across 0.5, 1, 1.25, and the intertemporal elasticity of substitution in consumption between 1 and 0.40 [i.e. \( \gamma = 0, -1.5 \)]. These parameters yield the following steady states (which are independent of \( \gamma \)):

\(^{19}\) See e.g. Cooley (1995) which summarizes the conventional parameters employed in the real business cycle literature.
\(^{20}\) In the United States total government spending is approximately 20% of GDP, of which about 20% (i.e. 4% of GDP) is devoted to public investment. Setting \( \eta = 0.20 \) falls well within the range of evidence cited by Gramlich (1994).
Table 2: Initial Steady States

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{K}/\bar{Y}$</th>
<th>$\tilde{C}/\bar{Y}$</th>
<th>$\tilde{l}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.5$</td>
<td>1.72</td>
<td>0.87</td>
<td>0.703</td>
</tr>
<tr>
<td>$\sigma = 1$</td>
<td>2.96</td>
<td>0.81</td>
<td>0.727</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>3.89</td>
<td>0.77</td>
<td>0.749</td>
</tr>
</tbody>
</table>

Given its prominent role in macrodynamics in general, and in modern growth theory in particular, the Cobb-Douglas function ($\sigma = 1$) serves as a plausible benchmark but our range is chosen in the light of recent empirical evidence based on different data sets and covers the range of plausible estimates.\(^{21}\)

The exercise that is performed is to introduce two increases in $g$: (i) a “moderate” increase, from 0.04 to 0.05 which implies a 25% increase in $G$ from its initial level, and (ii) a “large” increase, from 0.04 to 0.08, i.e. a 100% increase from the initial level of $G$.\(^ {22}\) The resulting dynamic paths are computed first for the linearized system and then using the nonlinear reverse shooting algorithm. As noted in Section 2, the system has only one state variable, $K$. Thus the linearized system, being saddlepoint-stable, has only one negative eigenvalue and one independent jump variable.

The results for the errors in the time paths of key variables and in the corresponding welfare changes are summarized in Tables 3 and 4 respectively. Since both the initial and final steady states are calculated accurately, the errors due to linearization are associated only with the transitional path. Table 3 summarizes the errors from linearization for a moderate increase in $g$ (from 0.04 to 0.05), while Table 4 reports the corresponding errors for a large increase in $g$ (from 0.04 to 0.08). As

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\(^{21}\) As justification for the Cobb-Douglas, Berndt’s (1976) early comprehensive study is often cited. For the preferred methods of estimation, using superior data, he finds estimates of the elasticity of substitution to range from around 0.8 to 1.2. However, recent authors have argued that the treatment of technological change has biased the estimates toward unity, and that modifying the econometric specification leads to significantly lower estimates of the elasticity, in the range 0.5-0.7, thus rejecting the Cobb-Douglas specification; see e.g. Antrás (2004), Klump, McAdam, and Willman (2007). Duffy and Papageorgiou (2000) estimate the elasticity of substitution using cross-sectional data and find that the Cobb-Douglas production function is an inadequate representation of technology across countries. Their evidence suggests that the elasticity of substitution exceeds unity for rich countries, but is less than unity for developing countries.

\(^ {22}\) While these rate of public investment are arbitrary, the base rate of 4% is chosen as being close to the recent US experience. We have also introduced comparable increases in the rates of public investment, starting from an initial point nearer the optimum (which can be computed), to see whether the errors from linearization were sensitive to the deviation from the optimum. However, the base value of $g$ had little impact on our findings.
noted, the errors in the time paths of the linearized solutions (Table 3 and 4, Panel A) are expressed as a percentage of the initial value of the respective variable. Panels B in Tables 3 and 4 report the welfare gains as the percentage increase in the permanent consumption flow [over the pre-shock level] that delivers the same utility over time $T$ as does the increase in $g$. The details of the welfare calculation are reported in the Appendix. Since the changes in welfare have been transformed into equivalent percentage changes in consumption flows, the errors from linearization are more appropriately measured by (18’) which expresses them as percentage point errors. However, to illustrate the corresponding magnitudes of the errors expressed as actual percentages, we also report the latter.

As both Tables 3 and 4 yield the same pattern, we shall focus primarily on the former (a moderate increase in government spending). We first consider the case $\sigma = 1$, and $\gamma = -1.5$, which serves as a commonly employed benchmark. This yields the eigenvalue $\lambda = -0.054$, implying an asymptotic speed of convergence of 5.4%, generally consistent with the updated empirical evidence. For the moderate fiscal expansion in Table 3, the errors are all less than 0.06% of their respective initial values, while when $g$ is doubled (Table 4), they increase by about a factor of 11.

In the case of income, the linear approximation overstates its initial increase by 0.056% of its pre-shock level, which is equivalent to 1.25% error vis-a-vis the true increase, while, the overstatement of 0.655 in Table 4 is equivalent to a 4.23% error. Over time these decline steadily.

The initial increase in consumption, together with decline in leisure, leads to an initial welfare increase equivalent to 1.836% of the pre-shock consumption level. Because of the overprediction of consumption the linearized model predicts a 1.894% increase in welfare, which is a 0.058 percentage point error, equivalent to a 3.17% error. Accumulated over time these errors decline to 0.02 percentage points (3.657% vs. 3.637%), which is approximately a 0.57% error. If $g$ is doubled from 0.04 to 0.08, these errors increase from 0.632 percentage points, on impact, and to

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23 Early benchmark estimates of speeds of convergence were estimated to be around 2-3% per annum; see e.g. Mankiw et al (1992), Barro and Sala-i-Martin (1992). Subsequent research has questioned the accuracy of these original estimates, suggesting that they ignore a number of econometric issues, as a result of which they are downwardly biased. Once one controls for these factors, the estimates of the convergence rates both increase and become more sensitive to the time period, the set of countries and their stages of development; see e.g. Islam (1995), Caselli, Esquivel, and Lefort (1996), Evans (1997), and Temple (1998).
0.233 percentage points, over time. Thus for the benchmark case, the errors committed by linearization do not seem to be misleading and should be acceptable. Even if the rate of expenditure is doubled, the estimated accumulated welfare effects from linearization overstate the true effect by only 1.23% (13.064 vs. 12.905).

Other interesting patterns that can be identified include the following:

(i) More rapid convergence (e.g. when $\sigma = 0.5$, and $\gamma = 0$) is associated with larger initial errors, but these also decline more rapidly over time. However, when there is slower convergence (e.g. $\sigma = 1.25$ and $\gamma = 0$), the errors from linearization are much more persistent. The initial errors are most pronounced for consumption (the jump variable). Comparing consumption for the logarithmic utility function, we see that the initial error is 0.101 for $\sigma = 0.5$ ($\lambda = -0.313$) and only 0.048 if $\sigma = 1$ ($\lambda = -0.135$). After 5 periods the errors in the former case will have dropped to 0.015 as compared to 0.017 in the latter case.

(ii) In contrast, the error in the sluggish variable, $K$, is zero initially (since jumps in the state variable are ruled out), and increases gradually for the first few periods along the transitional path, before declining.

(iii) The speed of convergence declines (and therefore the errors from linearization increase) as the elasticity of substitution in production increases. In contrast, the speed of convergence increases (and the errors from linearization decline) as the intertemporal elasticity of substitution increases.

(iv) Both consumption and leisure are over-estimated when $\sigma$ is small. For larger $\sigma$, leisure tends to be under-estimated and consumption over-estimated, with the resulting errors in computing welfare being partially offsetting.

(v) For moderate increases in $g$, the percentage point errors in measuring the welfare gains are relatively small, often of the order of 0.07 in the short run and reduced to 0.01 when taken over the entire infinite horizon. These errors increase by about a factor of 10 when $g$ is doubled. Because of the small true changes in welfare in the short run, the percentage error committed by linearizing can be substantially larger. For example, for logarithmic utility and Cobb-Douglas production function linearization over-estimates the short-run welfare gain of
0.136 percentage points by 0.077, which is equivalent to an error of over 56% (of the true gain), though this fall dramatically over time.

The overall message from these simulations for the flow specification of government expenditure is that the errors committed using linearization are generally tolerable, and in no cases lead to seriously misleading qualitative conclusions, particularly if one focuses on the longer run.

5.2. The “Stock” Model: Two State Variables Case

We now turn to the stock version of the model outlined in Section 3. As in the flow case, we present results for a moderate and a large increase in $g$ both for the linearized system and the actual nonlinear system. The parameterization of the model is same as that of the flow model. The only additional parameter is the rate of depreciation of public capital or infrastructure, $\delta_G$, which is set to 0.035, following its calibration as in Turnovsky (2004).

For the sensitivity analysis, like the flow case, we allow the elasticity of substitution in production, $\sigma$, to vary across 0.5, 1, 1.25, and the intertemporal elasticity of substitution in consumption between 1 and 0.4 [i.e. $\gamma = 0,-1.5$]. In all these cases, the initial steady state for the private variables is identical to that of the flow model (see Table 2). In addition, the ratio of public to private capital is 0.66, 0.39, 0.29, corresponding to the three elasticity measures $\sigma = 0.5, 1, 1.25$.

Further sensitivity analysis is also conducted for different degrees of external effects of infrastructure on production by allowing $\eta$ to vary across 0.05 and 0.40 for the benchmark case with $\sigma = 1, \gamma = -1.5$.24 It may be noted that the system now has two state variables: the stocks of private and public capital. Accordingly, the dynamic system has two negative eigenvalues as required for saddle-path stability. For the benchmark case they are $\lambda_1 = -0.069$ and $\lambda_2 = -0.025$.

5.2.1. A Moderate Increase in Government Spending

We begin by presenting the results for the moderate increase in public investment in infrastructure, which corresponds to an increase in $g$ from its initial value of 0.04 to 0.05. Table 5A

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24 This does not affect the steady-state values listed in Table 2 for the benchmark case.
and 5B present the errors from linearization in predicting key variables, and the welfare gains, respectively, and are directly analogous to Table 3A and 3B. In addition, Table 5C presents some sensitivity results for welfare as the importance of public capital in production is varied.

The time paths for the key variables for the benchmark case \( \sigma = 1, \gamma = -1.5, \eta = 0.2 \) are illustrated in Fig. 1, where the solid line illustrates the “correct” nonlinear solution and the dashed line the linearized approximation.\(^{25}\) Focusing first on the solid line, we see that an increase in the rate of government investment immediately shifts resources away from the private sector, causing an initial decline in the rate of private investment. The increase in public investment immediately raises the productivity of labor causing an initial increase in labor supply, leading to an initial increase in output. However this is insufficient to meet the economy-wide demand and consumption immediately falls. Over time, as public capital is accumulated, productivity of private investment increases, leading to an overall expansion of the economy, including employment and consumption. This is the gist of Figs 1.1-1.6.

It is also evident from Fig. 1 that while the linearized approximation yields a generally similar pattern, it over-predicts the short-run responses of all key variables. To see why this occurs it suffices to focus on the case of inelastic supply, when the following second-order approximations to the returns on capital and labor obtain

\[
\begin{align*}
    r &\equiv F_k = F_k^* + F_{kk}^* (K - K^*) + \frac{1}{2} F_{kkk}^* (K - K^*)^2 \\
    w &\equiv F_L = F_L^* + F_{Lk}^* (K - K^*) + \frac{1}{2} F_{Lkk}^* (K - K^*)^2
\end{align*}
\]

where for the Cobb-Douglas, as well as for plausible values of the elasticity of substitution, \( F_{kkk} > 0, F_{Lkk} < 0 \). Thus, by ignoring the second order (and higher) terms, the linear approximation first overstates the return to labor, as the capital stock increases. As a consequence, the linearization over-predicts initial labor supply and therefore initial output. It also understates the decline in the rate of return on capital as capital is accumulated, which also leads to a consumption response that is systematically higher than in the nonlinear model, where the latter also accounts for the second order

\(^{25}\) We did not provide corresponding figures for the “flow” model, since the time paths track those of the nonlinear model very closely.
Comparing Tables 3A, 3B and Tables 5A and 5B, the errors, which are in comparable units, are substantially larger than in the flow (one-state variable) case. Consider the case of income and the benchmark parameters ( $\sigma = 1, \gamma = -1.5, \eta = 0.2$ ). The initial error of 0.76%, while still small, is nearly 14 times larger than that for the flow model, and its more gradual decline reflects the slower rate of convergence. Even the direction of impact effects (instantaneous response to the shock) is wrong in some cases, implying a qualitatively erroneous response. For example, in Fig. 1, the nonlinear model predicts an initial 0.28% downward jump in consumption in response to the government spending shock, whereas the linear model predicts a 0.05% upward jump.

But from Fig. 1 two things are apparent. Although the linearized approximation appears to track the true system reasonably closely, there are significant errors that are committed at the initial stages following the policy expansion. And, occurring early in the dynamic evolution means that they weigh heavily in the welfare calculations. Figs 1.7 and 1.8 illustrate the time paths for the instantaneous welfare and the accumulated welfare paths, associated with the nonlinear model and its linear approximation. On impact, the nonlinear model implies a welfare loss of 0.550% of initial consumption, whereas the linear model implies a smaller 0.313% loss. This is an error of 0.237 percentage points that is more or less sustained throughout the entire time path, since overall the linearized model overpredicts the long-run intertemporal gain by 0.265 percentage points (1.048% vs. 0.783%). This is significantly larger than are the corresponding errors in the flow model, both measured in percentage point terms, and as percentages of the changes themselves.

Table 5A and 5B summarizes the errors from linearization for other combinations of the structural parameters, $\sigma$ and $\gamma$. These are all substantially higher than are the corresponding errors for the flow model. As $\sigma$ increases and $\gamma$ decreases, the errors from linearization increase in magnitude, since the underlying model becomes more non-linear in these cases. In addition, sensitivity analysis conducted for elasticity of substitution in production and elasticity of intertemporal substitution [but not reported] show that

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26 The over-estimate of consumption together with the under-estimate of leisure in the linearized model are partially offsetting in the assessment of the error in estimating welfare.
(i) as in the flow case, higher elasticity of substitution reduces the speed of convergence whereas higher elasticity of intertemporal substitution increases it. Further, higher speed of convergence is typically associated with larger initial errors that die down faster when compared to the case with lower speed of convergence.

(ii) However, errors in key variables are much larger than that in the flow case, irrespective of the values of $\sigma$ and $\gamma$.

An important source of nonlinearity in the model is the externality arising from public capital accumulation, parameterized by $\eta$. As this externality increases, nonlinear effects become more important. Table 5C reports the welfare changes implied by the linearized and non-linear solutions in the benchmark case for three values of $\eta$: 0.05, 0.2, and 0.4, along with the implied errors over different time horizons. Fig. 2 illustrates the transition dynamics of the model when $\eta = 0.4$. Compared to the benchmark case ($\eta = 0.2$) in Fig. 1, it can easily be seen that when the externality is large, the errors from linearization last longer during transition for all the key variables. In particular, now there are perceptible errors even in the path of public capital unlike in the benchmark case, when it was tracked closely. More significantly, the linearized approximation now yields a qualitatively incorrect prediction about the impact effect of the government expansion on labor supply. While the true model implies an instantaneous downward jump in labor supply, the linear approximation now implies an upward jump. This error arises because the linear approximation also ignores terms such as $F_{\delta gKg} < 0$, which now become important and reinforces the overstatement of the real wage rate discussed in equation (19b). These contrasting predictions have important implications for the calculation of welfare changes. As Table 5C indicates, the welfare prediction of the linearized model deteriorates with an increase in $\eta$. For example, when $\eta = 0.4$, the percentage point errors in welfare increase by a factor of 3 in the short run and by almost 4 in the long run.

5.2.2. A Large Increase in Government Spending

The errors from linearization again magnify for a large increase in $g$, from 0.04 to 0.08 (a 100% increase). Comparing Table 6A with 5A, they increase by a factor of around 8 in the short run
which is slightly less than for the flow version. Fig. 3 illustrates the adjustment paths of the key variables from which the following conclusions, robust to variations in $\sigma$ and $\gamma$, emerge:

(i) A comparison with Figure 2 (a moderate increase in $g$) reveals that errors are much larger in this case. Once again, the linearized approximation over-predicts the short-run and transitional response of all variables (consumption, labor supply, public and private capital, and output). The errors are more persistent and last for a longer time during the transition.

(ii) The qualitative instantaneous impact of the shock on consumption as predicted by the linearized model is misleading, when compared with the response in the non-linear model. While the linearized model predicts an upward jump in consumption in response to the fiscal shock, the non-linear model actually implies that consumption falls on impact. Further, the linearized model fails to predict a sharp capital decumulation in the short run in response to the strong consumption smoothing motive.

The above errors generated by the linearized model along the transition path are also reflected in errors in the calculation of the accumulated welfare gains over different time horizons, as shown in Table 6A and 6B. In percentage terms, the errors are much larger than those in Table 5. In the short run, the linear approximation grossly underestimates the short-run losses and seriously overstates the long-run welfare gains. For example, whereas the true model predicts a long-run welfare increase of 1.939% the linear approximation overstates this by 2.249 percentage points.

Looking at Fig. 1.8 we see that for between period 6 and 38 the linear approximation will suggest positive accumulated welfare gains, whereas in fact the true nonlinear model indicates accumulated losses during that time frame. In some cases the errors are seriously misleading. For example, the linear approximation suggests that doubling the rate of government investment will lead to accumulated welfare gains of just over 1% over the first 20 years, whereas in fact the true model suggests an almost 1.4% welfare loss. In absolute terms, the errors in welfare predictions of the linearized model hover around 2 percent of initial consumption, in contrast to more modest errors of 0.2-0.3 percent for the moderate increase case. These are huge errors given the fact that the policy generates welfare gains equivalent to 1-4 percent of initial consumption.

Finally, we may note that the effects of the higher externality associated with public capital,
As in Fig. 2, the errors from linearization persist longer during transition for all the key variables, including the path of public capital, and the impact effect on labor supply is in the wrong direction (relative to the non-linear model). The effect of the variation in $\eta$ on the calculation of intertemporal welfare is reported in Table 6C. As in Table 5C, an increase in $\eta$ increases the absolute error from linearization in accumulated welfare changes over all time horizons. However, now the errors are quantitatively much larger and more persistent. For the high value of $\eta = 0.40$, the linearized solution over-predicts accumulated welfare at different time horizons by 8-9 percent of initial consumption, compared to the 2-2.5 percent over-prediction for the case where $\eta = 0.2$. Even for $\eta = 0.05$, the errors are quite significant, though much smaller. Interestingly, for such a low $\eta$, a large increase in $g$ reduces overall welfare as it takes public investment away from its optimal level.

### 5.3. Speed of Convergence

The speed at which a dynamic system approaches its steady-state equilibrium is another important aspect in the evaluation of the impact of public policies. For example, if convergence to the steady state is rapid, then one can evaluate public policies with respect to their long-run impact on the economy. On the other hand, if convergence is slow, then a significant part of the dynamic adjustment takes place far away from the steady-state, which in turn underscores the study of the transitional aspects of public policies. In this context, a natural question that arises is: do linearized systems over-predict or under-predict the speed of convergence relative to those implied by non-linear systems?

Following Barro and Sala-i-Martin (2004), we consider the following measure of the speed of the transitional path, called $\beta$-convergence:27

\[ \nu(t) \equiv -\frac{(\dot{x}(t) - \ddot{x})}{(x(t) - \ddot{x})} \]

This measure is quantitatively similar to the $\beta$-convergence measure we use for our analysis.

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27 Several measures of the speed of convergence can be found in the literature. A frequently used metric is the stable eigenvalue of the linearized system, as in Ortiguera and Santos (1997). While this measure is suitable for one-dimensional transition paths, it is not appropriate for systems with more than one state variable. Another measure, proposed by Eicher and Turnovsky (1999), characterizes the time-varying nature of the speed of convergence in higher order models, and is defined as: $\nu(t) \equiv -\frac{(\dot{x}(t) - \ddot{x})}{(x(t) - \ddot{x})}$. This measure is quantitatively similar to the $\beta$-convergence measure we use for our analysis.
where $x(t)$ is a generic state variable, such as the capital stock. The speed of convergence $\nu_x$ therefore measures the rate at which the growth of the variable $x$ slows down as it increases proportionately (i.e., approaches its steady-state equilibrium). For our analysis, this speed of convergence is calculated and compared for both the linearized and non-linear models in the cases where government expenditure enters production as a flow or a stock.

Figure 4 depicts the speed of convergence implied by the linearized and non-linear models for both the flow and stock cases and its sensitivity to the magnitude of policy changes and structural parameters. The following patterns emerge from this comparison:

(i) In the flow model (Panel I), the speed of convergence of the linearized transition path is always higher than that implied by the non-linear path. However, the magnitude of this over-prediction is quite small, and the two measures converge quite rapidly following a particular shock. Thus, consistent with our previous analysis, the errors from linearization does not seem to be too serious when the dynamic system is characterized by a quick transitional adjustment (e.g., one state variable).

(ii) In the stock model, the errors in the calculation of the speed of convergence from linearization are much more significant, both qualitatively and quantitatively. First of all, the initial jump in the speed of convergence on the impact of a shock is substantially larger in the linearized model relative to the non-linear model. Second, in three out of the four shocks considered (a large increase in $g$, a fall in $\sigma$, and an increase in $\gamma$), the short-run impact on the speed of convergence is exactly the opposite in the linearized and non-linear models: while the linearized path implies an instantaneous increase in the speed of convergence, the non-linear path implies an instantaneous decline. Third, during transition, the speed of convergence implied by the linearized adjustment path is consistently higher than that
implied by the non-linear path. This difference lasts for an average of 15-20 years following each shock, after which there is a tendency for both measures to converge.  

6. Temporary Shocks

While a permanent increase in government spending serves as the natural benchmark illustration of the potential errors from linearization, governments frequently increase investment as a temporary source of stimulus. It is therefore also important to consider the sensitivity of these errors to the duration of a shock. In this section we briefly summarize the effect of temporary fiscal shocks on the predictions of the linearized and non-linear solutions to the models developed in sections 2 and 3. We consider both moderate and large spending shocks as in the previous sections, but assume that these shocks now last for a 5-year period, after which government spending is restored to its initial, pre-shock level.

6.1. Temporary Shocks in the “Flow” Model

As in the case of a permanent shock, the errors from linearizing the flow model with public investment in the benchmark case ($\sigma = 1$) are minimal, both for the moderate and the large government spending shock. This leads us to infer that the duration of a policy shock does not generate serious errors from linearization in the flow model with a Cobb-Douglas technology. However, when the elasticity of substitution in production increases from $\sigma = 1$ to $\sigma = 1.25$, the errors from linearization start increasing. One consequence of the higher elasticity of substitution is that it slows down the speed of convergence to the pre-shock steady-state (as the temporary shock ends, the economy must transition back to its pre-shock saddle path). As discussed in Section 5, a slower speed of convergence can generate larger errors from linearization. Overall, our experiments suggest that the flow model does a reasonable job of approximating the “true” dynamic path of a

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28 As mentioned in the introduction, Papageorgiou and Perez-Sebastian (2007) show that the asymptotic speed of convergence implied by a linearized growth model does a poor job of replicating the actual convergence paths of countries like Japan and South Korea. Their analysis, therefore, calls for a complete characterization of the transition path, rather than just a study of the asymptotic dynamics (implied by linearization). Our comparison of the linearized and non-linear convergence paths is thus consistent with their findings.
system, irrespective of whether the shock is permanent or temporary, although if the elasticity of substitution increases above unity, the errors start to increase.\(^{29}\)

In the case of a permanent shock, we have noted previously that the linearized model, in some cases, incorrectly predicted the instantaneous responses of consumption and leisure, which eventually contributed to an over-prediction of intertemporal welfare changes across steady states. However, in the case of a temporary shock two differences arise. First, the direction of the instantaneous responses are correctly predicted by the linearized model. Second, instead of over-predicting consumption on impact, linearization now overpredicts consumption at the time of withdrawal of the shock. This is due to the fact while the economy jumps to the stable manifold on impact for the permanent case, it does so for the temporary case only at the time of the withdrawal of the shock after following the unstable path for the duration of the shock.

6.2. Temporary Shocks in the “Stock” Model

As demonstrated in Section 5, introducing public capital as a stock can lead to both qualitative and quantitative errors from linearization that are substantially larger than those in the flow model. This pattern is magnified in the case of a temporary shock, as shown in Fig. 5.\(^{30}\) Even with a moderate temporary shock, the errors from linearization are larger than for the corresponding permanent shock. This is because the introduction of a second state variable (public capital) slows down the speed of the transition path, both when the shock is in effect (along the unstable trajectory) and when it has ended (along the stable trajectory). This leads to a compounding of the errors. Whereas linearization committed qualitative errors in assessing the short-run response of consumption only for large permanent shocks, this occurs even for moderate temporary shocks.

Comparing the linearized and non-linear solutions for the stock model with temporary and permanent shocks, one interesting difference arises. While linearized model still overpredicts consumption, in the temporary case, labor supply is underpredicted in contrast to the permanent case. As a result, impact of errors in consumption and leisure on welfare is mutually reinforcing for

\(^{29}\) We do not illustrate these results graphically, but the figures are available from the authors on request.

\(^{30}\) For temporary shocks for stock case, we use the forward shooting technique developed in Atolia and Buffie (2008).
temporary shock. In particular, errors in welfare assessment are (likely?) to be larger for the temporary shock.

7. Conclusions

In this paper we have addressed an important practical question with respect to the effects of government policy on the dynamic evolution of an economy: how serious are the errors in employing conventional linearized solution methods? To an important degree, the answer to this question depends upon the policymaker’s tolerance for error. To address this, we have considered two versions of a Ramsey type growth model. In the first, government expenditure appears as a flow and the economy is described by a first order dynamic system, whereas in the second, it is introduced as a stock, thereby rendering the dynamics second order.

Overall, our results suggest that in the flow model, the errors committed by linearization are generally small and do not generate seriously misleading results. Even if one doubles the rate of government spending – a large quantitative change – the errors of linearization are tolerable. In the short run they are typically less than 1% of the initial value of the respective variable and they decline quite rapidly over time. The corresponding errors for welfare gains (or losses) are also tolerable, being less than one percentage point in the short run to around 0.2-0.3 percentage points in terms of the long-run accumulated welfare gains. Given the model’s parsimony, this should be a tolerable error.

The stock specification is more difficult. The errors generally are of the of the order of 6 times those of the flow model. This is because the higher order system generates slower adjustments and these are associated with larger errors. In the short run, the linear approximation may yield incorrect qualitative predictions. This is particularly true as the importance of government capital in production increases, and the size of the change in government investment increases. In these cases the linearized model can yield misleading conclusions insofar as the short run is concerned. The error in estimating welfare changes caused by a change in g is 33.8% in the benchmark stock case for a moderate shock, and rises to 116% when the shock is large. For some runs the errors were as high as 244%.
However, provided one proceeds judiciously, our analysis suggests that linearization may still be useful. In this regard if: (i) one restricts oneself to “moderate policy changes” as in Table 5, and (ii) focuses on the longer run when the linear model tracks the true path more closely, the errors are not seriously misleading. Thus, if one uses the model to provide a “bottom line” estimate of the long-run benefits of the policy change, errors of the order summarized in Table 5B should be tolerable. But with the higher order model, linearization may be less reliable for short-run analysis, when the errors can be substantial, both quantitatively and qualitatively. In that case, one may employ non-linear solution procedures such as forward and reverse shooting, the application of which has become more practical with the rapid increase in computing capacity.

As we have noted at several points, the key factors determining the errors involved in linearization are the initial responses in consumption and labor supply (leisure), which in turn are critical determinants of intertemporal welfare changes. This has important consequences for the effect on the dynamics of wealth and income distribution as well. Elsewhere, Turnovsky and García-Peñalosa (2008) have shown that the dynamics of wealth and income distribution, following some structural change, depends critically upon \((l(0)-\bar{l})\). The accuracy with which the linear system estimates this quantity is therefore important for assessing the evolution of wealth and income distribution. An important extension of this paper is to examine this issue.
Table 3: The “Flow” Model (one state variable case)

Moderate increase in $g$: 0.04 to 0.05

A. Errors at different points of time ($T$)

<table>
<thead>
<tr>
<th>$T$</th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_1 = -0.313$</td>
<td>$\lambda_1 = -0.140$</td>
</tr>
<tr>
<td>$\sigma = 0.50$</td>
<td>$c$</td>
<td>.101</td>
</tr>
<tr>
<td></td>
<td>$l$</td>
<td>.038</td>
</tr>
<tr>
<td></td>
<td>$k$</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>$y$</td>
<td>.160</td>
</tr>
<tr>
<td>$\sigma = 1.00$</td>
<td>$\lambda_1 = -0.135$</td>
<td>$\lambda_1 = -0.054$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>.048</td>
</tr>
<tr>
<td></td>
<td>$l$</td>
<td>-.044</td>
</tr>
<tr>
<td></td>
<td>$k$</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>$y$</td>
<td>.057</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>$\lambda_1 = -0.084$</td>
<td>$\lambda_1 = -0.033$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>.040</td>
</tr>
<tr>
<td></td>
<td>$l$</td>
<td>-.063</td>
</tr>
<tr>
<td></td>
<td>$k$</td>
<td>.000</td>
</tr>
<tr>
<td></td>
<td>$y$</td>
<td>.044</td>
</tr>
</tbody>
</table>

B. Welfare gain on impact and in the long run (% of initial consumption)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 0$</td>
<td>$T = \infty$</td>
</tr>
<tr>
<td>$\sigma = 0.50$</td>
<td>Linear</td>
<td>0.235</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>0.164</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.071</td>
</tr>
<tr>
<td></td>
<td>%Error</td>
<td>43.32%</td>
</tr>
<tr>
<td>$\sigma = 1.00$</td>
<td>Linear</td>
<td>0.213</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>0.136</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.077</td>
</tr>
<tr>
<td></td>
<td>%Error</td>
<td>56.40%</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>Linear</td>
<td>-0.050</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>-0.126</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.076</td>
</tr>
<tr>
<td></td>
<td>%Error</td>
<td>-60.52%</td>
</tr>
</tbody>
</table>
Table 4: The “Flow” Model (One state variable case)

Large increase in $g$: 0.04 to 0.08

A. Errors at different points of time ($T$)

<table>
<thead>
<tr>
<th></th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.50$</td>
<td>$\lambda_1 = -0.309$</td>
<td>$\lambda_1 = -0.132$</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>$l$</td>
</tr>
<tr>
<td>$T$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$\sigma = 1.00$</td>
<td>$\lambda_1 = -0.134$</td>
<td>$\lambda_1 = -0.054$</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>$\lambda_1 = -0.084$</td>
<td>$\lambda_1 = -0.033$</td>
</tr>
</tbody>
</table>

B. Welfare gain on impact and in the long run (% of initial consumption)

<table>
<thead>
<tr>
<th></th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.50$</td>
<td>Linear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>%Error</td>
</tr>
<tr>
<td>$\sigma = 1.00$</td>
<td>Linear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>%Error</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>Linear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>$T = 0$</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>%Error</td>
</tr>
</tbody>
</table>
Table 5: The “Stock” Model (Two-State variable Case)

Moderate increase in $g$: 0.04 to 0.05

A. Errors at different points of time ($T$)

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$T$</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma=0.50$</td>
<td>$\gamma=0$</td>
<td>$\lambda_1=-0.315$; $\lambda_2=-0.028$</td>
<td>.245</td>
<td>.398</td>
<td>.459</td>
<td>.411</td>
<td>.260</td>
<td>.061</td>
<td>.333</td>
<td>.389</td>
<td>.423</td>
<td>.407</td>
<td>.296</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_1=-0.150$; $\lambda_2=-0.027$</td>
<td>.149</td>
<td>.068</td>
<td>.013</td>
<td>-.013</td>
<td>-.016</td>
<td>-.012</td>
<td>.057</td>
<td>.047</td>
<td>.035</td>
<td>.022</td>
<td>.007</td>
</tr>
<tr>
<td>$\sigma=1.00$</td>
<td>$\gamma=0$</td>
<td>$\lambda_1=-0.147$; $\lambda_2=-0.027$</td>
<td>.548</td>
<td>.523</td>
<td>.468</td>
<td>.370</td>
<td>.223</td>
<td>.049</td>
<td>.757</td>
<td>.728</td>
<td>.675</td>
<td>.577</td>
<td>.399</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_1=-0.069$; $\lambda_2=-0.027$</td>
<td>.245</td>
<td>.328</td>
<td>.390</td>
<td>.400</td>
<td>.308</td>
<td>.089</td>
<td>.332</td>
<td>.353</td>
<td>.369</td>
<td>.370</td>
<td>.320</td>
</tr>
<tr>
<td>$\sigma=1.25$</td>
<td>$\gamma=0$</td>
<td>$\lambda_1=-0.099$; $\lambda_2=-0.026$</td>
<td>.573</td>
<td>.506</td>
<td>.422</td>
<td>.312</td>
<td>.171</td>
<td>.025</td>
<td>.760</td>
<td>.706</td>
<td>.633</td>
<td>.528</td>
<td>.365</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_1=-0.050$; $\lambda_2=-0.021$</td>
<td>.221</td>
<td>.283</td>
<td>.339</td>
<td>.368</td>
<td>.319</td>
<td>.118</td>
<td>.305</td>
<td>.317</td>
<td>.328</td>
<td>.331</td>
<td>.305</td>
</tr>
</tbody>
</table>

B. Welfare gain on impact and in the long run (% of initial consumption)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\gamma$</th>
<th>$T=0$</th>
<th>$T=\infty$</th>
<th>$T=0$</th>
<th>$T=\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma=0.50$</td>
<td>$\gamma=0$</td>
<td>Linear</td>
<td>-1.034</td>
<td>1.485</td>
<td>-0.475</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nonlinear</td>
<td>-1.168</td>
<td>1.218</td>
<td>-0.765</td>
</tr>
<tr>
<td></td>
<td></td>
<td>%pt Error</td>
<td>0.134</td>
<td>0.267</td>
<td>0.290</td>
</tr>
<tr>
<td></td>
<td></td>
<td>% Error</td>
<td>11.47%</td>
<td>21.92%</td>
<td>37.91%</td>
</tr>
<tr>
<td>$\sigma=1.00$</td>
<td>$\gamma=0$</td>
<td>Linear</td>
<td>-0.839</td>
<td>1.201</td>
<td>-0.313</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nonlinear</td>
<td>-0.925</td>
<td>0.944</td>
<td>-0.550</td>
</tr>
<tr>
<td></td>
<td></td>
<td>%pt Error</td>
<td>0.086</td>
<td>0.257</td>
<td>0.237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>% Error</td>
<td>9.30%</td>
<td>27.22%</td>
<td>43.09%</td>
</tr>
<tr>
<td>$\sigma=1.25$</td>
<td>$\gamma=0$</td>
<td>Linear</td>
<td>-0.850</td>
<td>0.943</td>
<td>-0.351</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nonlinear</td>
<td>-0.904</td>
<td>0.696</td>
<td>-0.550</td>
</tr>
<tr>
<td></td>
<td></td>
<td>%pt Error</td>
<td>0.054</td>
<td>0.247</td>
<td>0.199</td>
</tr>
<tr>
<td></td>
<td></td>
<td>% Error</td>
<td>5.97%</td>
<td>35.49%</td>
<td>36.18%</td>
</tr>
</tbody>
</table>
C. Sensitivity analysis for welfare gain

<table>
<thead>
<tr>
<th>η</th>
<th>Linear</th>
<th>Nonlinear</th>
<th>%pt Error</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>T = 0</td>
<td>T = ∞</td>
<td></td>
<td></td>
</tr>
<tr>
<td>η = 0.05</td>
<td>-1.080</td>
<td>-0.605</td>
<td>0.044</td>
<td>3.91%</td>
</tr>
<tr>
<td></td>
<td>-1.124</td>
<td>-0.650</td>
<td>0.045</td>
<td>-6.92%</td>
</tr>
<tr>
<td>η = 0.20</td>
<td>-0.313</td>
<td>1.048</td>
<td>0.237</td>
<td>43.09%</td>
</tr>
<tr>
<td></td>
<td>-0.550</td>
<td>0.783</td>
<td>0.265</td>
<td>33.84%</td>
</tr>
<tr>
<td>η = 0.40</td>
<td>0.983</td>
<td>3.884</td>
<td>0.759</td>
<td>338.8%</td>
</tr>
<tr>
<td></td>
<td>0.224</td>
<td>2.927</td>
<td>0.957</td>
<td>32.70%</td>
</tr>
</tbody>
</table>
Table 6: The “Stock” Model (Two-State variable Case)

A. Errors at different points of time ($T$)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T=0$</td>
<td>$T=2$</td>
</tr>
<tr>
<td>$\sigma=0.50$</td>
<td>$\lambda_1 = -0.315; \lambda_2 = -0.028$</td>
<td>$\lambda_1 = -0.315; \lambda_2 = -0.028$</td>
</tr>
<tr>
<td>$c$</td>
<td>2.15</td>
<td>3.49</td>
</tr>
<tr>
<td>$l$</td>
<td>1.34</td>
<td>.636</td>
</tr>
<tr>
<td>$k$</td>
<td>.000</td>
<td>2.90</td>
</tr>
<tr>
<td>$y$</td>
<td>4.79</td>
<td>4.69</td>
</tr>
<tr>
<td>$\sigma=1.00$</td>
<td>$\lambda_1 = -0.147; \lambda_2 = -0.027$</td>
<td>$\lambda_1 = -0.147; \lambda_2 = -0.027$</td>
</tr>
<tr>
<td>$c$</td>
<td>2.12</td>
<td>2.84</td>
</tr>
<tr>
<td>$l$</td>
<td>2.07</td>
<td>1.30</td>
</tr>
<tr>
<td>$k$</td>
<td>.000</td>
<td>1.82</td>
</tr>
<tr>
<td>$y$</td>
<td>4.43</td>
<td>4.23</td>
</tr>
<tr>
<td>$\sigma=1.25$</td>
<td>$\lambda_1 = -0.099; \lambda_2 = -0.026$</td>
<td>$\lambda_1 = -0.099; \lambda_2 = -0.026$</td>
</tr>
<tr>
<td>$c$</td>
<td>1.88</td>
<td>2.43</td>
</tr>
<tr>
<td>$l$</td>
<td>2.39</td>
<td>1.64</td>
</tr>
<tr>
<td>$k$</td>
<td>.000</td>
<td>1.32</td>
</tr>
<tr>
<td>$y$</td>
<td>3.87</td>
<td>3.70</td>
</tr>
</tbody>
</table>

B. Welfare gain on impact and in the long run (% of initial consumption)

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 0$</td>
<td>$T = \infty$</td>
</tr>
<tr>
<td>$\sigma = 0.50$</td>
<td>Linear</td>
<td>-3.686</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>1.130</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>23.46%</td>
</tr>
<tr>
<td>$\sigma = 1.00$</td>
<td>Linear</td>
<td>-3.215</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>-3.948</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.733</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>18.57%</td>
</tr>
<tr>
<td>$\sigma = 1.25$</td>
<td>Linear</td>
<td>-3.449</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>-3.915</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.466</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>11.90%</td>
</tr>
</tbody>
</table>
C. Sensitivity analysis for welfare gain

<table>
<thead>
<tr>
<th></th>
<th>$T = 0$</th>
<th>$T = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta = 0.05$</td>
<td>Linear</td>
<td>-4.272</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>-4.621</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>0.349</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>7.55%</td>
</tr>
<tr>
<td>$\eta = 0.20$</td>
<td>Linear</td>
<td>-0.538</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>-2.553</td>
</tr>
<tr>
<td></td>
<td>%pt</td>
<td>2.015</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>78.93%</td>
</tr>
<tr>
<td>$\eta = 0.40$</td>
<td>Linear</td>
<td>7.645</td>
</tr>
<tr>
<td></td>
<td>Nonlinear</td>
<td>0.221</td>
</tr>
<tr>
<td></td>
<td>%pt Error</td>
<td>7.424</td>
</tr>
<tr>
<td></td>
<td>% Error</td>
<td>3359%</td>
</tr>
</tbody>
</table>
Figure 1: The “Stock” Model - A moderate increase in $g$: 0.04 to 0.05

Time paths for variables for benchmark case ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.2$)
Figure 2: The “Stock” Model - A moderate increase in $g$: 0.04 to 0.05

Time path for variables when $\eta$ is large ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.4$)
Figure 3: The “Stock” Model - A large increase in $g$: 0.04 to 0.08

$\gamma = -1.5, \sigma = 1, \eta = 0.2$

Linearized Model:  
Non-Linear Model:  

Figure 4: Speed of Convergence in the “Flow” and “Stock” Models

I. Flow Model

II. Stock Model

Panel A: Large increase in $g$

Panel B: Moderate increase in $g$

Panel C: Lower $\sigma$

Panel D: Higher $\gamma$

Linearized Model: 
Non-Linear Model:
Figure 5: The “Stock” Model - A moderate temporary 5 year increase in $g$: 0.04 to 0.05

Time paths for variables for benchmark case ($\gamma = -1.5$, $\sigma = 1$, $\eta = 0.2$)

Phase Diagram

Private Capital Stock

Public Capital Stock

Output

Consumption

Labor Supply

Instantaneous Welfare

Accumulated Welfare

Linearized Model: ————
Non-Linear Model: ————
Appendix

In this paper, we report accumulated welfare gains over different time horizons, $T$. We quantify the welfare gains as the (equivalent) percent increase in consumption over initial consumption that delivers same utility over lifetime as the increase in $g$.

To compute this measure over time horizon $T$, we first compute the total utility over the time horizon after implementation of the increase in $g$

$$W_T(C(t),l(t)) = \int_0^T \frac{1}{\gamma} [C(t)]^\gamma e^{-\beta t} dt,$$

where $C(t)$ and $l(t)$ are time paths of consumption and leisure based on the relevant model, linear or nonlinear. The corresponding total utility in absence of increase in $g$ is

$$W_T(C_o,l_o) = \int_0^T \frac{1}{\gamma} [C_o]^{\gamma} e^{-\beta t} dt.$$

We seek to find $\zeta$ such that

$$W_T(\zeta C_o,l_o) = W_T(C(t),l(t)),$$

which implies that the (equivalent) percent increase in consumption over initial consumption as reported in Panels B of Tables 3 and 4

$$\zeta - 1 = \left[ \frac{W_T(C(t),l(t))}{W_T(C_o,l_o)} \right]^{\frac{1}{\gamma}} - 1.$$

It may be mentioned that this consumption equivalent measure of welfare change takes into account the utility gain or loss arising out of change in leisure due to increase in $g$. 

A1
References


Arrow, K.J. and M. Kurz, 1970, Public Investment, the Rate of Return, and Optimal Fiscal Policy, Johns Hopkins Press, Baltimore, MD.


Cambridge, MA.

