An OLG Model of Tax Evasion with Public Capital*

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Abstract

The paper presents a dynamic overlapping generations model of tax evasion where government revenue is used to provide public capital. It establishes existence and uniqueness of the competitive equilibrium and presents a detailed characterization of its dynamics. An increase in the probability of being caught, and the penal tax rate reduces tax evasion along the entire equilibrium path - a result that holds in the existing models in the literature across steady states. In the extended small open economy model with tariff on imported capital, distortions due to tax evasion wipe out the gains from the tariff reform.

Keywords: Tax Evasion, Public Investment, Dynamic Analysis, Existence.

JEL Classification: H26, H41, C61, C62.

1 Introduction

Tax evasion is an important activity in all economies. Rey [21] in one of the earliest published estimates of tax evasion found that evasion of Italian General Sales Tax was 52.46% of the actual yield. Other similar studies include Gutmann [16] and Feige [14]. For India, Acharya [1] estimated that only 53.3% of total assessable income was declared.

Not all tax evaders are caught, but if caught, they pay taxes at a higher, penal tax rate. Thus, the framework of choice under uncertainty applies very naturally to the decision to evade taxes. Allingham and Sandmo [3] is a seminal contribution to the theory of tax evasion which models decision to evade taxes in this framework where agents maximize expected utility. Other contributions include Srinivasan [23] and Yitzhaki [27]. In these models there is no tax evasion if the expected penal tax rate is at least as high as the statutory tax rate. Further, an increase in the probability of being caught or the penal tax rate lowers tax evasion.

Subsequently Cross and Shaw [7] have analyzed tax evasion and tax avoidance jointly and Landskroner, Paroush, and Swary [17] look at tax evasion in more general models of choice of under uncertainty with additional risky assets. In these extensions basic comparative statics results, mentioned above, survive so long as there is decreasing absolute risk aversion. However, all these papers use static partial equilibrium models with exogenously given income.

The model in the paper generalizes these models in three different directions by using general equilibrium analysis with productive public expenditure in a dynamic setting. I extend the basic comparative static results of

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these papers to this dynamic general equilibrium model. I use the overlapping generations framework introduced by Samuelson [22]. This model has been used extensively for the analysis of alternative government economic policies and important extensions include Cass and Yaari [6] and Diamond [9].

I work with an overlapping generations economy where each generation lives for two periods. The government imposes a tax on the labor income which agents can evade. The government audits, at zero cost, a fraction, $p$, of the returns and those who are caught pay tax at a higher penal tax rate. Though tax collection is costless the government cannot change $p$, by using more resources. This simplifies the analysis. The government uses its revenues productively and its expenditure decisions affect the equilibrium of the economy. I assume that the government makes investment in the public capital, which enters the production function of the firms. I follow Barro [4] by introducing public investment as an externality in the production function of the firms. However, the paper does not look at an endogenous growth model.\footnote{Detailed theoretical analysis of one-sector models with public capital is done in Glomm and Ravikumar [15]. Turnovsky [25] discusses role of public capital in a small open economy with endogenous growth.}

For very general production function and preferences, the model has a unique competitive equilibrium. The results from existing static models mentioned earlier hold in this model. The paper shows that these results, in fact, hold along the entire equilibrium path of the economy for a wide class of production function and preferences. The results also generalize to economies that are open, though small, with respect to world trade but can not borrow or lend abroad.\footnote{If a country is small and it can borrow and lend abroad, the interest rate become exogenous and hence the results generalize quite readily.}

With Cobb-Douglas production function and CRRA preferences the transition dynamics of this economy is similar to a neo-classical economy, and capital and consumption converge monotonically to their long-run levels. Further, I also find that savings and private capital accumulation decline with fall in tax evasion. This ties in with the existing literature on precautionary savings. The savings provide ‘self-insurance’ against the eventuality of being caught if agent evades taxes. A reduction in tax evasion reduces this need for self-insurance.

To present an application of the model, I consider an open economy where government levies tariff on the imported capital. The economy is small with respect to trade and hence, in absence of distortions, free trade is the optimal policy. But, it cannot borrow or lend abroad and hence the real interest rate is determined endogenously. The issue of trade liberalization has been at the center stage over the past decade and half but the developing countries have only very reluctantly agreed to liberalize their trade regimes. In this open economy model, due to tax evasion, a reduction in/ removal of tariff does not yield a Pareto improvement.

The remaining portion of the paper is organized as follows. Section 2 outlines the model. Section 3 contains the results on the existence and uniqueness of the equilibrium. Section 4 presents and solves the simplified model. Section 5 contains the extension to an open economy. The welfare effects of a trade reform are analyzed in section 6. Section 7 concludes.
2 The Model

I consider a two-period overlapping generations economy with constant population. The population of each generation has measure 1 and each individual has measure zero. Each agent is endowed with one unit of labor when young which he supplies inelastically.

The agents have time additive separable (von Neumann-Morgenstern) utility function. Each individual derives utility from consumption in both periods. Agents have no bequest motive. The government makes investment in the public capital, $G$, that is non-rival and augments the productivity of each firm. The firms are identical with production function $F(K, G, L)$.

There are standard assumptions on the utility function and the production function:

Assumption 1. (i) The per period utility function $u(.)$ is strictly increasing, strictly concave, twice continuously differentiable and satisfies Inada conditions.

(ii) The production function has constant returns of scale and diminishing marginal product in private capital and labor. In addition, the private capital and the public capital are gross complements (i.e. $F_{KG} > 0$)

As population of each generation has measure 1, in equilibrium, economy wide aggregates or averages, as the case may be, of most variables coincide with their individual level values.

Notation. In general variable corresponding to individuals are denoted by small letters and (corresponding) economy wide variables by (corresponding) capital letters.

Household’s Problem. The representative agent of generation $t$ has labor income $w_t$ when young which he earns by supplying one unit of labor. Thus, $w_t$ is also the wage rate. In period $t+1$, when old, he derives income from the capital saved in period $t$.

The government levies a tax at the rate $\tau_{it}$ on labor income earned in period $t$ which agents can evade. It audits an exogenously given fraction $p$ of the returns. When a return is audited tax evasion is detected with probability 1. When caught, agents pay penal tax on the unreported labor income in period $t$ itself at the higher rate $\tau_{it}^p$.

Therefore, on receiving his income, along with his consumption-saving decision, an agent of generation $t$ decides the fraction of the labor income on which to evade taxes. He cannot diversify away the risk of tax evasion. Though, at the time of making this decision, he knows the probability of being caught and the penal tax rate $\tau_{it}^p$. Based on this information, when young, an agent of generation $t$ decides, the fraction of labor income, $x_t$, to hide and the savings, $s_t$, for the next period. After taking this decision, he learns in the same period if he
has been caught. If caught, he pays penal taxes from his savings. Thus, an agent of generation \( t \) accumulates \( k_{1,t+1} \) amount of capital if he is not caught and \( k_{2,t+1} \) otherwise.

So, the representative agent of generation \( t \) solves the following optimization problem to maximize his lifetime utility

\[
\max_{c_{yt},s_t,x_t,c_{ot+1}^1,c_{ot+1}^2} \quad u(c_{yt}) + \beta (1 - p) u(c_{ot+1}^1) + \beta p u(c_{ot+1}^2)
\]

subject to

\[
c_{yt} + s_t \leq [x_t + (1 - x_t)(1 - \tau_t)]w_t + j_t, \tag{2}
\]

\[
c_{ot+1}^1 \leq [r_{t+1} + (1 - \delta)]k_{1,t+1}, \tag{3}
\]

\[
c_{ot+1}^2 \leq [r_{t+1} + (1 - \delta)]k_{2,t+1}, \tag{4}
\]

\[
k_{1,t+1} = s_t, \tag{5}
\]

\[
k_{2,t+2} = s_t - \tau_{t+1}^p x_t w_t, \tag{6}
\]

where \( c_{yt} \) is the consumption of an agent of generation \( t \) when he is young, \( c_{ot+1}^1 \) is his consumption when old (i.e. in period \( t + 1 \)) if he is not caught evading taxes, \( c_{ot+1}^2 \) is the consumption when old if he is caught evading taxes, and \( r_{t+1} \) is the (gross) return on capital from period \( t \) to period \( t + 1 \). As the utility function is strictly increasing in consumption in each period, all budget constraints hold with equality in equilibrium.

The dynamics of the model is very simple. All young are born equal and hence are homogeneous \textit{ex ante}. All agent’s in a generation, solve the same problem and decide to evade tax on the same fraction of income and save the same amount for the next period. However, \textit{ex post} they split into two groups: a fraction \( p \) who are caught evading taxes form a group which has lower per capita capital and the remaining individuals form the other group with higher capital per capita. Thus, at any point in time there are only these three types of agents in the economy and the stock of capital is owned by the old. A measure \( p \) of individuals have one amount of capital and remaining individuals have another amount of capital. The aggregate capital at the beginning of the next period is the weighted average of the capital of these two groups.

\textbf{Firms’ Problem.} The firms hire capital and labor and maximize profits in each period. Thus, their optimization problem is:

\[
\max_{K_t, L_t} \pi_t = F(G_t, K_t, L_t) - r_t K_t - w_t L_t. \tag{7}
\]

As there are constant returns to scale in the private factors, without loss of generality, we can consider only one firm that employs all the factors of the economy.
Government’s Budget Constraint. The government’s budget constraint for period $t$ is

$$R_t = G_{t+1} - (1 - \delta_G)G_t + J_t = [\tau_{it} (1 - X_t) W_t + \tau^p_{it} pX_t W_t],$$

(8)

where $X_t$ is the average fraction of income not reported by the agents of generation $t$, $\delta_G$ is the rate of depreciation of the public capital, $J_t$ is the total transfer made to the generation $t$ in period $t$, $W_t$ is aggregate wage income of the young in period $t$, $G_{t+1}$ is the investment in public capital that enters production function of the firms in period $t+1$, and $R_t$ is the total government revenue from labor income tax in period $t$. The government revenues in excess of public investment are rebated lump-sum to the current young from whom they are collected. Thus, there are no intergenerational transfers as is the case with the poor developing countries.\(^3\)

Competitive Equilibrium I now define a competitive equilibrium for this economy.

Definition. A competitive equilibrium for this economy is a sequence $\{c_{yt}, c^1_{ot+1}, c^2_{ot+1}, x_t, s_t, k_{1,t}, k_{2,t}, l_t, r_t, w_t, K_t, L_t, G_t, j_t, J_t, \tau_{it}, \tau^p_{it}\}$ such that:

1. For every $t$, given $\{r_{t+1}, w_t, j_t, \tau_{it}, \tau^p_{it}\}$, $\{c_{yt}, c^1_{ot+1}, c^2_{ot+1}, x_t, s_t\}$ solves the optimization problem (1) for the agent of generation $t$.

2. For every $t$, given $\{r_t, w_t, G_t\}$, $\{K_t, L_t\}$ maximizes profit of the firms as in (7).

3. For every $t$, given $\{c_{yt}, c^1_{ot+1}, c^2_{ot+1}, x_t, s_t, k_{1,t+1}, k_{2,t+1}, r_{t+1}, w_t, K_t, L_t, G_t\}$, government policy $\{G_{t+1}, J_t, \tau_{it}, \tau^p_{it}\}$ satisfies government’s budget constraint (8).

4. All markets clear and aggregate and individual variables are consistent for all $t$. Specifically markets for capital, labor and output clear:

$$C_{yt} = c_{yt}, \quad C^1_{ot-1} = (1 - p)c^1_{ot-1}, \quad C^2_{ot-1} = pc^2_{ot-1},$$

$$K_{t+1} = (1 - p)k_{1,t+1} + pk_{2,t+1} + S_t - pr^p_{it} X_t W_t,$$

$$L_t = l_t = 1, \quad J_t = j_t, \quad S_t = s_t, \quad X_t = x_t, \quad W_t = w_t,$$

$$C_{yt} + C^1_{ot-1} + C^2_{ot-1} + (K_{t+1} - (1 - \delta_k)K_t) + (G_{t+1} - (1 - \delta_G)G_t) = Y_t.$$  

3 Solving the Model

The firms’ problem (7) gives the following first-order conditions:

$$K_t : \quad r_t = F_{K,t},$$

$$L_t : \quad w_t = F_{L,t}.$$
where, in general, $F_{t,t} = \partial F (G_t, K_t, L_t) / \partial I, I = G, K, \text{ or } L.$

Besides consumption and savings, agents also choose the fraction of labor income not to report. The Kuhn- Tucker conditions for maximization are

$$
\begin{align*}
    s_t & : u'(c_{gt}) = \beta \left[ (1 - p)u'(c_{ot+1}^1) + pu'(c_{ot+1}^2) \right] \left( F_{K,t+1} + (1 - \delta_k) \right), \\
    x_t & : \tau_{it} u'(c_{gt}) \leq \beta p \tau_{it} u'(c_{ot+1}^2) \left( F_{K,t+1} + (1 - \delta_k) \right) \text{ if } x_t = 0, \\
    & : \tau_{it} u'(c_{gt}) = \beta p \tau_{it} u'(c_{ot+1}^2) \left( F_{K,t+1} + (1 - \delta_k) \right) \text{ if } 0 \leq x_t \leq 1, \\
    & : \tau_{it} u'(c_{gt}) \geq \beta p \tau_{it} u'(c_{ot+1}^2) \left( F_{K,t+1} + (1 - \delta_k) \right) \text{ if } x_t = 1.
\end{align*}
$$

In the first order condition for $x_t$ the left hand side is the marginal benefit of evading taxes on labor income and right hand side is the marginal cost. Thus, at corner solution corresponding to no tax evasion, i.e., $x_t = 0$, the marginal cost is higher than the marginal benefit.

The aggregate private capital and government capital for the next period are given by

$$
\begin{align*}
    K_{t+1} &= s_t - pr_{it}^p x_t w_t, \\
    G_{t+1} &= [\tau_{it} - (\tau_{it} - pr_{it}^p) x_t] w_t + (1 - \delta_G) G_t - j_t.
\end{align*}
$$

Here I have replaced the aggregate variables by individual variables using the equilibrium conditions. Thus, given the initial capital stocks ($K_t, G_t$), for the interior solution, (9), (10), (11), and (12), are 4 equations in 4 unknowns, namely, current decisions of the agents, $s_t$ and $x_t$, and the stock of private and public capital, $K_{t+1}$ and $G_{t+1}$. It is clear that (11) defines $K_{t+1}$ as an increasing function of $s_t$ and as a decreasing function of $x_t$ which I represent as $K^t(s_t^+, x_t^-)$. Similarly (12) defines $G_{t+1}$ as $G^t(x_t^-)$. Here $^t$ denotes the dependence on the government policy in period $t$. Substitution for $K_{t+1}$ and $G_{t+1}$ in (9) and (10) as $K^t(s_t^+, x_t^-)$ and $G^t(x_t^-)$ yields two equations in $s_t$ and $x_t$.

In general (10) is not satisfied as equality. There is a corner solution under the following condition:

**Proposition 1.** Under Assumption 1, if the expected penal rate is at least as high as the statutory tax rate on labor income, i.e., if $p r_{it}^p \geq \tau_{it}$, there exists a unique equilibrium with no tax evasion.

**Proof.** Assume $x_t = 0$. Then $c_{ot+1}^1 = c_{ot+1}^2$ and using (9) we get

$$
u'(c_{gt}) = \beta u'(c_{ot+1}^2) \left( F_{K,t+1} + (1 - \delta_k) \right) \leq \beta \frac{pr_{it}^p}{\tau_{it}} u'(c_{ot+1}^2) \left( F_{K,t+1} + (1 - \delta_k) \right),$$

where last inequality follows from the fact that $pr_{it}^p \geq \tau_{it}$. Thus, (10) also hold for corner solution with $x_t = 0$ which is thus a solution to the agent’s problem.
Suppose $x_t > 0$ is a solution. Then as $c^1_{ot+1} > c^2_{ot+1}$

\[
u'(c_{yt}) = \beta \left[ (1 - p)u'(c^1_{ot+1}) + pu'(c^2_{ot+1}) \right] \left[ F_{K,t+1} + (1 - \delta_t) \right]
\]
\[
< \beta u'(c^2_{ot+1}) \left[ F_{K,t+1} + (1 - \delta_t) \right] \leq \beta \frac{p_y}{\tau_t} u'(c^2_{ot+1}) \left[ F_{K,t+1} + (1 - \delta_t) \right],
\]

which violates Kuhn-Tucker conditions in (10) for $x_t > 0$.

This is the result obtained in Allingham and Sandmo [3], Yitzhaki [27] and Landskroner et al. [17] among others and is a straightforward implication of the risk-aversion of the agents and the expected utility framework. It is also quite intuitive and follows immediately from the risk aversion of the agents and the use of expected utility framework. When $p\tau^F_{it} = \tau_{it}$, expected tax liability is same whether the agent evades or pay taxes but due to risk aversion, he prefers to pay taxes. Though the expected tax liability is same, due to concavity of the utility function, the certainty equivalent of the expected loss, if agent chooses to evade taxes is larger.

### 3.1 Existence and Uniqueness of the Competitive Equilibrium

I now establish the existence and uniqueness of the competitive equilibrium when there is tax evasion in the economy. This happens when

**Assumption 2.** $p\tau^F_{it} < \tau_{it}$.

For further analysis I also need following additional assumptions.

**Assumption 3.** Both types of the capital depreciate fully in each period.

**Assumption 4.**

\[
(i) \quad 1 + \frac{F_{K,t+1}}{s_tF_{KK,t+1}} \leq \frac{1}{\sigma} \leq 1,
\]

\[
(ii) \quad 1 + \frac{F_{K,t+1}}{s_t \left[ F_{KK,t+1} - F_{KG,t+1} \frac{\tau_{it}-p\tau^F_{it}}{\tau^F_{it}} \right]} \leq \frac{1}{\sigma} \leq 1,
\]

where $\sigma = \text{coefficient of relative risk aversion}$.

As we shall see, Assumption 4(i) ensures existence of the competitive equilibrium. It restricts the curvature of the production function relative to the curvature of the utility function. In the absence of tax evasion it would imply $-K_{t+1}F_{KK,t+1}/F_{K,t+1} < \frac{\sigma}{\sigma - 1}$, a condition which holds for the Cobb-Douglas production function and CRRA preferences. The presence of tax evasion creates a wedge between savings and capital accumulation which yields this modified condition. Under a more restricted condition on $\sigma$ given in Assumption 4(ii) the competitive equilibrium is also unique. Assumption 4 is always satisfied for log preferences. Assumption 3 is made only for the simplification of the analysis.
The presence of externality in the form of public investment can result in non-existence and/or multiplicity of equilibria. The problem can arise if public capital is so productive that an increase in tax on private capital raises marginal product of capital indirectly through an increase in the stock public capital. When \( \sigma > 1 \) the effect of public capital on marginal product of capital does not lead to non-existence and hence \( F_{KG} \) terms does not appear in Assumption 4(i).

Under Assumption 1 – 3 and using the firms’ first order conditions, the consumption when young and old are

\[
c_yt = [(1 - \tau_d) + x_t \tau_{it}] w_t + j_t - s_t
\]  \hspace{1cm} (13)

\[
c_{ot+1}^1 = s_t F_{K,t+1}
\]  \hspace{1cm} (14)

\[
c_{ot+1}^2 = (s_t - \tau^p_{it} x_t w_t) F_{K,t+1},
\]  \hspace{1cm} (15)

and the equations governing the evolution of the two capital stocks and the first order conditions become

\[
s_t: u'(c_{yt}) = \beta \left[ (1 - p) u'(c_{ot+1}^1) + p u'(c_{ot+1}^2) \right] F_{K,t+1}, \hspace{1cm} (A)
\]

\[
x_t: u'(c_{yt}) = \frac{\beta \tau^p_{it}}{\tau_{it}} u'(c_{ot+1}^1) F_{K,t+1} \hspace{1cm} \text{if} \hspace{0.5cm} 0 \leq x_t \leq 1,
\]  \hspace{1cm} (B)

\[
x_t: u'(c_{yt}) \geq \frac{\beta \tau^p_{it}}{\tau_{it}} u'(c_{ot+1}^2) F_{K,t+1} \hspace{1cm} \text{if} \hspace{0.5cm} x_t = 1.
\]

\[
K_{t+1} = s_t - p \tau^p_{it} x_t w_t,
\]  \hspace{1cm} (16)

\[
G_{t+1} = [\tau_{it} - (\tau_{it} - p \tau^p_{it}) x_t] w_t - j_t.
\]  \hspace{1cm} (17)

To prove the existence and the uniqueness of the equilibrium, I proceed by proving a sequence of lemmas. For brevity of exposition, the proofs of lemmas are relegated to the Appendix.

**Lemma 1.** Under Assumptions 1 - 3, and 4 (i) and given the initial capital stocks \( (K_t, G_t) \), (A) defines \( s_t \) as an upward sloping continuous function of \( x_t \), i.e. it defines \( s_t = S^A_t(x^+_t) \).

**Lemma 2.** Under Assumptions 1-3, and 4(i) and given the initial capital stocks \( (K_t, G_t) \), (B) defines \( s_t \) as an upward sloping continuous function of \( x_t \), i.e. it defines \( s_t = S^B_t(x^+_t) \).

**Lemma 3.** Let Assumptions 1 - 3, and 4 (i) hold and the initial capital stocks be given. If \( s^{A0}_t \) and \( s^{B0}_t \) solve \( (A) \) and \( (B) \) respectively when \( x_t = 0 \) then \( s^{B0}_t < s^{A0}_t \).

Now, I establish the existence and the uniqueness of the competitive equilibrium for this economy.

**Proposition 2.** Under Assumptions 1 - 3, and 4 (i), there exists a competitive equilibrium for this economy.

**Proof.** Given any policy \( (\tau_{it}, \tau^p_{it}) \) and \( p \) such that Assumption 2 is satisfied, Lemma 3 shows \( s^{B0}_t < s^{A0}_t \). Thus, the curve representing \( S^B_t(x^+_t) \) defined in Lemma 2 lies below that representing \( S^A_t(x^+_t) \) defined in Lemma 1 at \( x_t = 0 \) (See Figure 2.)
Let $s_t^{A1}$ and $s_t^{B1}$ solve (A) and (B) respectively when $x_t = 1$. If there does not exist an equilibrium with $0 < x_t < 1$ then clearly $s_t^{B1} \leq s_t^{A1}$ and I assert

Claim 1. If $s_t^{A1} \geq s_t^{B1}$ then there exists an equilibrium with $x_t = 1$.

Proof of Claim 1. The corresponding equilibrium value of $s_t$ solves (A) and hence equals $s_t^{A1}$. We know $s_t^{B1}$ solves (B) for $x_t = 1$. Once again note that $LHS(B)$ is increasing in $s_t$ and $RHS(B)$ is decreasing in $s_t$. Thus, for every $s_t \geq s_t^{B1}$, $LHS(B) \geq RHS(B)$ and hence also for $s_t = s_t^{A1}$. So, the Kuhn-Tucker condition for the corner solution with $x_t = 1$ is satisfied. Hence Claim 1 and Proposition 2 are proved. ■ ■

![Diagram](image)

**Figure 1:**

**Proposition 3.** Under Assumptions 1 - 3, and 4 (ii), the competitive equilibrium is unique.

**Proof.** I first outline the argument of the proof. I claim that if $S_t^A(x_t^\dagger)$ and $S_t^B(x_t^\dagger)$ intersect then at the point of intersection $dS_t^A/dx_t < dS_t^B/dx_t$ (See Figure 1.) Given this claim and given $s_t^{B0} < s_t^{A0}$ there are only two possibilities. First possibility is that the two curves intersect once for $0 < x_t < 1$. Then there exists an interior equilibrium. As $s_t^{A1} < s_t^{B1}$ in this case, a simple argument proves the converse of Claim 1 in proof of Proposition 2, and thus, there can not be an equilibrium with $x_t = 1$. Hence the interior equilibrium is unique.

In the second case, $S_t^A(x_t^\dagger)$ lies strictly above $S_t^B(x_t^\dagger)$ for $0 \leq x_t < 1$, in which case there is no interior equilibrium. But, as $s_t^{A1} \leq s_t^{B1}$, there exists a unique equilibrium with $x_t = 1$ by virtue of Claim 1 in the Proof of Proposition 2. This proves uniqueness of the equilibrium except for the following:

Claim 2. If the curves $S_t^A(x_t^\dagger)$ and $S_t^B(x_t^\dagger)$ intersect for $0 < x_t < 1$ then

$$\frac{dS_t^A}{dx_t} < \frac{dS_t^B}{dx_t},$$

at the point of intersection.
Proof of Claim 2. Implicit differentiation of (B) with respect to \( x_t \) gives:

\[
\frac{dS^t_B}{dx_t} = \tau_{il}w_t \left( \frac{h_k}{F_{K,t+1}} \frac{p_{ij}^p}{\tau_{il}} - \frac{u''(c_{yt})}{u'(c_{yt})} \right) > 0,
\]

in which by Assumption 1 and Lemma 1A both the numerator and the denominator are positive.\(^4\) Assumption 4(ii) gives a sufficient condition for the fraction on the \( \text{RHS} \) of (18) to be less than \( s_t/(\tau_{il}x_tw_t) \).

Similarly, implicit differentiation of (A) with respect to \( x_t \) gives

\[
\frac{dS^t_A}{dx_t} = \tau_{il}w_t \left( \frac{h_k}{F_{K,t+1}} \frac{p_{ij}^p}{\tau_{il}} - \frac{u''(c_{yt})}{u'(c_{yt})} - \frac{\beta(1-p)\sigma F_{K,t+1}u'(c_{yt+1})}{\sigma F_{K,t+1}u'(c_{yt})} \right) > 0.
\]

Note that compared to \( dS^t_B/dx_t \) the numerator and the denominator in the expression for \( dS^t_A/dx_t \) each has an additional term being subtracted from it.\(^5\) The term being subtracted from the numerator is \( s_t/(\tau_{il}x_tw_t) \) times the term being subtracted from the denominator. But the fraction on \( \text{RHS} \) of (18) is less than \( s_t/(\tau_{il}x_tw_t) \), and thus a simple algebraic fact implies that \( dS^t_A/dx_t < dS^t_B/dx_t \). Hence, Claim 2 and Proposition 3 are proved. \( \blacksquare \)

Further characterization of the model in such generality is not plausible and hence I turn to a simplified model for which closed form analytical solutions can be worked out. Then I provide results from numerical simulations of some more general specifications. Prior to that the next subsection gives an example where competitive equilibrium does not exist.

3.2 Non-existence of Equilibrium - An Example

Assumption 4(i) is a sufficient and not a necessary condition for existence of the competitive equilibrium. This section presents an example in which violation of this assumption leads to non-existence of equilibrium.

Consider an economy with CRRA preferences and \( \sigma > 1 \) and the production function

\[
F(K_t, G_t, L_t) = A(\min[aG_t, K_t])^a L_t^{1-a},
\]

\( a > 0, \quad 0 < \alpha < 1. \)

There are constant returns to scale and \( F_K > 0 \) so long as \( aG_t > K_t \) in equilibrium. But public investment is determined by government policy. If government policy is such that public capital stock is too low then \( F_K \) becomes zero and Assumption 4(i) is violated and the economy does not have an equilibrium.

Mathematically, under this condition, \( K_t = aG_t \), and hence (13), (14), (15), (A), (B), (16), and (17) are 7 equations in 6 unknowns, namely, \( c_{yt}, c_{yt+1}, c_{yt+1}^2, s_t, x_t \) and \( G_t \) and in general a solution is not guaranteed to exist.

\(^4\)See Appendix.

\(^5\)The intersection of two curves yields an equilibrium, hence the value of each endogenous variable in both (18) and (19) are identical and the two equations can be compared.
It is also possible to construct examples with $\sigma < 1$, for which competitive equilibrium does not exist. It happens when $\sigma$ is very low and $F_{KG}$ is very high. Formally, the last two inequalities in Lemma 1A can not be derived from the preceding two under these conditions.

4 A Simplified Model

In order further characterize the behavior of the model, I solve a simplified version of the model characterized by:

Assumption 1a. (i) The per period utility function is $u(c) = \log(c)$.

(ii)

$$F(G, K, L) = (AG^\theta)K^\alpha L^{1-\alpha} \equiv Y,$$

where $0 < \alpha < 1, \theta > 0$, and $0 < \alpha + \theta < 1$.

Thus the preferences are logarithmic. The production function is Cobb-Douglas where public capital augments the total factor productivity and there are diminishing returns to the accumulable factors. The output per person of the generation that supplies labor is

$$y = f(K, G) = (AG^\theta)K^\alpha,$$

where $K$, $L$, and $G$ are economy-wide aggregate variables.

I solve the model under Assumption 1a, and 3. Assumption 4 is always satisfied with log-preferences. By Assumption 3, both public and private capital depreciate 100% in every period. These simplifications give useful insight into the dynamics of the model and help provide a detailed characterization of the equilibrium and the steady state. The government budget constraint continues to be given by (12). The firms’ problem (7) gives the following first-order conditions for an interior solution:

$$r_t = \frac{Y_t}{K_t}, \quad \text{and} \quad w_t = (1 - \alpha)\frac{Y_t}{L_t} = (1 - \alpha)Y_t.$$

The Kuhn-Tucker conditions for maximization are

\begin{align*}
    s_t : \frac{1}{c_{pt}} &= \beta(1 - p) + \frac{\beta p}{s_{t+1}} - \frac{\beta p r_{p}^{p}}{s_{t+1} - r_{p}^{p}x_{t}w_{t}}, \\
    x_t : \frac{\tau_{it}}{c_{pt}} &\leq \frac{\beta p r_{p}^{p}}{s_{t+1} - r_{p}^{p}x_{t}w_{t}} \quad \text{if } x_t = 0, \\
    \quad : \frac{\tau_{it}}{c_{pt}} = \frac{\beta p r_{p}^{p}}{s_{t+1} - r_{p}^{p}x_{t}w_{t}} \quad \text{if } 0 \leq x_t \leq 1, \\
    \quad : \frac{\tau_{it}}{c_{pt}} \geq \frac{\beta p r_{p}^{p}}{s_{t+1} - r_{p}^{p}x_{t}w_{t}} \quad \text{if } x_t = 1.
\end{align*}
I now impose equilibrium and solve (20) and (21) for the equilibrium ex ante\footnote{The ex post savings are reduced by the amount of the penal taxes paid if the agent is caught evading taxes.} individual savings function, \(s^t\), and individual ‘tax evasion’ function, \(x^t\), for the case when there is an interior solution which are 

\[
\begin{align*}
\tilde{s}^t(K_t, G_t) & = \frac{S_t}{Y_t} = \frac{s_t}{y_t} = (1 - p) \frac{\beta}{1 + \beta} \frac{\tau_{it}^p}{\tau_{it} - \tau_{it}} [(1 - \tau_{it})(1 - \alpha) + j_t/y_t], \quad (22) \\
x^t(K_t, G_t) & = X_t = x_t = \frac{1}{1 - \alpha} x_t = \frac{1}{1 - \alpha} \frac{\beta}{1 + \beta} \left[ \frac{1}{\tau_{it} - \tau_{it}} \right] [(1 - \tau_{it})(1 - \alpha) + j_t/y_t], \quad (23)
\end{align*}
\]

where \(y_t\) on the right hand side of (22) and (23) is a function of \((K_t, G_t)\). Since population size of each generation is normalized to 1 and only the current young, who are homogeneous ex ante, save and evade taxes, these functions also represent economy wide averages as well as aggregates. These functions give the unique solution to the agent’s decision problem which can then be used to determine the new values of (aggregate) state variables \(K_{t+1}\) and \(G_{t+1}\) using (16) and (17). The uniqueness of the corner solution follows from the general case discussed in the previous section.

When 

\[ p = p^* := \frac{\tau_{it}}{\tau_{it}} , \]

there is no tax evasion and the savings function reduces to 

\[ \tilde{s}_t = \tilde{s}^t(K_t, G_t) = \frac{\beta}{1 + \beta} [(1 - \tau_{it})(1 - \alpha) + j_t/y_t] \]

which is a standard result for one-sector overlapping generations growth models with log-preferences and full depreciation. Thus the model reduces to the standard model when government is able to set expected tax rate \(\tau_{it}^p\) equal to or above the threshold level \(\tau_{it}\). Thus, from now on I assume that Assumption 2 also holds so that there is tax evasion in the economy.

### 4.1 Dynamics of the Simplified Model

Analytical solution to the equilibrium path of the economy can be obtained for this case. I have already derived the ex ante savings function and the ‘tax evasion’ function. Further algebra gives the solution for the evolution of the state variables of the economy as 

\[
\begin{align*}
K^{t+1}(K_t, G_t) = K_{t+1} & = \frac{\beta}{1 + \beta} \left[ \frac{\tau_{it}^p}{\tau_{it}} p^2 + \frac{\tau_{it}^p}{\tau_{it} - \tau_{it}} (1 - p)^2 \right] [(1 - \tau_{it})W_t + J_t], \quad (25) \\
G^{t+1}(K_t, G_t) = G_{t+1} & = \left[ \tau_{it} + \frac{\beta \xi_t}{1 + \beta} (1 - \tau_{it}) \right] W_t - \left[ 1 - \frac{\beta \xi_t}{1 + \beta} \right] J_t, \quad (26)
\end{align*}
\]

with \(\xi_t = -\frac{(1 - p\tau_{it}^p/\tau_{it})^2}{\tau_{it}^p/\tau_{it} - 1}\) where \(\frac{\partial \xi_t}{\partial p} > 0, \ \frac{\partial \xi_t}{\partial \tau_{it}^p} > 0, \ \text{and} \ \frac{\partial \xi_t}{\partial \tau_{it}} < 0.\)

These equations describe the equilibrium dynamics of the economy given the initial stocks of capital \((K_t, G_t)\). Further characterization of the equilibrium is possible when the government policy is time-invariant. This is done in a later section of the paper.
Now, I turn to comparative dynamics of the model. First I present the results for the case when a change in the government policy or a parameter value results in a movement from an interior solution to a corner solution or vice versa.

**Proposition 4.** Under Assumptions 1a, and 2 - 4, and given \((\tau_{it}, j_t)\), ex ante savings and private capital accumulation is higher in presence of tax evasion.

**Proof.** When \(p \geq \tau_{it}/\tau_{it}^p\) there is no tax evasion and savings and capital accumulation are equal and given by (24). When \(p \leq \tau_{it}/\tau_{it}^p\) there is tax evasion. In this case

\[
(1 - p) \frac{\tau_{it}^p}{\tau_{it} - \tau_{it}^p} = (1 - p) \frac{1}{1 - \tau_{it}/\tau_{it}^p} \geq 1
\]

and hence comparison of (22) and (24) shows that ex ante saving is higher than savings without tax evasion. For the capital accumulation note that the critical point of the term within big brackets in (25), when differentiated with respect to \(p\), is

\[
p^* = \frac{\tau_{it}}{\tau_{it}^p}
\]

The second derivative at this critical point is:

\[
2 \left[ \frac{\tau_{it}^p}{\tau_{it}} + \frac{\tau_{it}^p}{\tau_{it}^p - \tau_{it}} \right] = 2 \left[ \frac{1}{p^*} + \frac{1}{1 - p^*} \right] > 0.
\]

This implies that \(p^*\) is a minimum and this term is 1 at this minimum. Further, (25) evaluated at \(p^*\) gives the capital accumulation in absence of tax evasion. Hence, in presence of tax evasion, the capital accumulation is higher if \(p\) is smaller. A similar argument applies when the term in the big bracket in (25), is differentiated with respect to \(\tau_{it}^p\). □

This result is quite intuitive. If there is no tax evasion, the agent saves only to accumulate capital. When he evades taxes he can not buy insurance against the event of being caught evading taxes. Hence, he increases saving as it also provides him ‘self insurance’ against the event of being caught. This is a familiar result in the literature on precautionary savings.

Now, I derive the comparative dynamics effects of change in \(p\), \(\tau_{it}^p\), and \(\tau_{it}\) on the extent of tax evasion in the economy. The literature on tax evasion has looked at effects of change in these parameters on tax evasion. For the interior equilibrium, we have:

**Proposition 5.** Under Assumption 1a, 2 - 4, and given that government keeps \(j_t/y_t\) unchanged

\[
\frac{\partial x_t}{\partial p} < 0, \quad \frac{\partial x_t}{\partial \tau_{it}^p} < 0, \quad \frac{\partial x_t}{\partial \tau_{it}} > 0,
\]

\(\tau_{jt}/y_t\) may be time varying.
private capital accumulation. Conversely, an increase in provide himself 'self insurance' against the event of being caught evading taxes. Higher saving leads to higher change (of the result in Proposition 4.2 Time-Invariant Government Policies in the Simplified Model

More can be said about the dynamics of the model when the government policies are time invariant in some appropriate sense. To further characterize equilibrium, I assume that all tax rates are constant. With regard to

\[
\frac{\partial s_t}{\partial p} < 0, \quad \frac{\partial s_t}{\partial \tau^p_{it}} < 0, \quad \frac{\partial s_t}{\tau_{it}} > 0,
\]
\[
\frac{\partial}{\partial p} \left( \frac{K_{t+1}}{Y_t} \right) < 0, \quad \frac{\partial}{\partial \tau^p_{it}} \left( \frac{K_{t+1}}{Y_t} \right) < 0,
\]
\[
\frac{\partial}{\partial p} \left( \frac{G_{t+1}}{Y_t} \right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \tau^p_{it}} \left( \frac{G_{t+1}}{Y_t} \right) > 0.
\]

**Proof.** From (23) it is evident that \( \partial x_t/\partial p < 0, \quad \partial x_t/\partial \tau^p_{it} < 0. \) Also \( x_t \) can be written as

\[
x_t = \frac{1}{1-\alpha \left( 1+\beta \right)} \left( 1 - \frac{p \tau^p_{it}}{\tau_{it}} \right) \left( \frac{1}{\tau^p_{it} - \tau_{it}} \right) (1 - \alpha) + \left( \frac{1}{\tau^p_{it} - \tau_{it}} \right) \frac{j_t}{y_t},
\]

which implies \( \partial x_t/\partial \tau_{it} > 0 \) as the terms in the three braces above are all increasing in \( \tau_{it} \). Equation (22) immediately yields \( \partial \tilde{s}_t/\partial p < 0, \ \partial \tilde{s}_t/\partial \tau^p_{it} < 0 \) and as \( \tilde{s}_t \) can be written as

\[
\tilde{s}_t = \frac{(1-p)}{(1+\beta)} \frac{\beta \tau^p_{it}}{\tau_{it}} \left( \frac{1}{\tau^p_{it} - \tau_{it}} \right) (1 - \alpha) + \left( \frac{1}{\tau^p_{it} - \tau_{it}} \right) \frac{j_t}{y_t},
\]

\( \partial \tilde{s}_t/\partial \tau_{it} > 0 \) as the terms in the two braces are increasing in \( \tau_{it} \). The proof of the Proposition 4 shows that \( \partial (K_{t+1}/Y_t)/\partial p < 0, \) and \( \partial (K_{t+1}/Y_t)/\partial \tau^p_{it} < 0. \) Lastly, from (26) I get \( \partial (G_{t+1}/Y_t)/\partial p > 0 \) and \( \partial (G_{t+1}/Y_t)/\partial \tau^p_{it} > 0 \) as \( \partial \xi/\partial p > 0, \) and \( \partial \xi/\partial \tau^p_{it} > 0 \) by virtue of (27). \]

Note that the current decisions of the agents, \( x_t \) and \( \tilde{s}_t \), are independent of the current state of the economy \( (K_t, \ G_t) \) and depend on the current policy \( (p) \) alone. So, do the current aggregate accumulation of the private and the public capital as fraction of the total output in period \( t \). Thus, equilibrium path of these variables is completely determined by the path of the government policy. For example, for any new path of penal tax rate \( \{\tau^p_{it}\} \) such that \( \tau^p_{it} \geq \tau_{it}, \forall \ t, \) the entire equilibrium path of \( x_t, \tilde{s}_t, \) and \( K_{t+1}/Y_t \) will be lower and that of \( G_{t+1}/Y_t \) will be higher than the original one. The same holds true for changes in \( p \). The argument also applied to effect of changes in \( \tau_{it} \) with respect to its effects on \( x_t, \) and \( \tilde{s}_t. \)

The fact that both \( x_t \) and \( \tilde{s}_t \) move together, as government policy (including \( p \)) changes, is a generalization of the result in Proposition 4. When an agent increases his extent of tax evasion, he also increases savings to provide himself ‘self insurance’ against the event of being caught evading taxes. Higher saving leads to higher private capital accumulation. Conversely, an increase in \( p \) and \( \tau^p_{it} \) causes private capital stock to be lower on the new equilibrium path due to reduction of tax evasion. The ratio of public capital to the output, on the other hand, rises because government revenue rises as a fraction of the output due to increase in the ‘effective tax rate’ on the labor income. The effective tax rate rises due to a fall in the fraction of the income on which tax is evaded and due to an increase in expected penal tax rate \( p \tau^p_{it} \) on the evaded income.

4.2 Time-Invariant Government Policies in the Simplified Model

More can be said about the dynamics of the model when the government policies are time invariant in some appropriate sense. To further characterize equilibrium, I assume that all tax rates are constant. With regard to

\^8Though \( p \) is assumed to be exogenous and constant in this paper, in the existing literature, it is assumed that government can change \( p \) by devoting more resources.
government transfers, $J_t$, time invariant policy can be characterized in two different but plausible ways.

**Assumption 5.** (i) $j_t/y_t$ is exogenous and constant over time.

(ii) $j_t/R_t$ ($= \zeta$) is exogenous and constant over time.

Under Assumption 5(i) the equations characterizing the dynamics are the same as in the previous section except for the fact that various tax rates and $j_t/y_t$ is unchanging over time. In some situations, it would be desirable to assume that transfers are a fixed fraction, $\zeta$, of its revenues in accordance with Assumption 5(ii). In this case, for the interior solution, the equations characterizing the dynamics become

\[ \dot{s}_{t+1} = \tilde{s}(K_t, G_t) = \frac{(1-p)\beta}{1+\beta(1-\xi)} \frac{\tau_i^p}{\tau_i \tau_t} \left[ (1-\tau_i) + \tau_i \zeta \right] (1-\alpha) \]  

\[ x_t = x(K_t, G_t) = \frac{\beta}{1+\beta(1-\xi)} \left[ 1 \right. \frac{p}{\tau_i} - \frac{p}{\tau_t} \left. \right] \frac{\left[ (1-\tau_i) + \tau_i \zeta \right]}{1-\tau_i/\tau_t} \]  

\[ K_{t+1} = K(K_t, G_t) = \frac{\tau_i^p}{\tau_i} \left[ \frac{(1-p)\beta}{\tau_i \tau_t} \right] \frac{\beta}{1+\beta(1-\xi)} \frac{W_t}{1-\tau_i/\tau_t} \]  

\[ G_{t+1} = G(K_t, G_t) = (1-\zeta) \frac{\left[ (1+\beta) \tau_i + \beta \xi (1-\tau_i) \right]}{1+\beta(1-\xi)} W_t. \]

### 4.2.1 Equilibrium Dynamics

The characterization of the competitive equilibrium is summarized in the following:

**Proposition 6.** Under Assumptions 1a, 3 and 4 and either Assumption 5(i) or Assumption 5(ii), the dynamics of the economy is characterized by:

1. **Constant level of tax evasion.**
2. **Constant ratio of stocks of the private and the public capital.**
3. **Constant ratios of private capital, public capital and ex ante savings to the output in every period.**
4. **Constant ratio of consumption of generation $t$ to the output in period $t$.**

**Proof.** I prove the Proposition under Assumption 5(ii). The proof with Assumption 5(i) is even easier and obvious from relevant equations.

First I consider the case when Assumption 2 holds. Statement 1 in the proposition is evident from (29). Statement 2 follows from (30) and (31). Equations (30), (31) and (28) imply Statement 3 as wage rate is a constant fraction of per capita output due to Cobb-Douglas production function.

With a little algebra, using the government budget constraint (12) and the equilibrium rule for accumulation of the public capital (31), one obtains:

\[ j_t = \zeta \frac{(1+\beta) \tau_i + \beta \xi (1-\tau_i)}{1+\beta(1-\xi)} w_t. \]  

(32)
Now using (2), (28) and (32) it easily follows that \( c_{yt}/y_t \) is constant. Similarly using (3), (5), and (28) it follows that \( c_{ot+1}/y_t \) is constant too. Then, equations (4), (6) and (28) show that \( c^2_{ot+1}/y_t \) is constant as well. This proves Statement 4 of the proposition.

When Assumption 2 does not hold the model reduces to the usual model without tax evasion where these results can be proved in a straightforward way. ■

The fact that the ratio of public and private capital is constant implies that the dynamics of the economy can be described by one state variable under the assumptions stated above. A number of features of the model lead to this outcome. Due to Assumption 5 and Cobb-Douglas production function, this ratio is completely determined by the current policy. When policy is time invariant this ratio is also constant over time.

4.2.2 Comparative Dynamics

Proposition 7. If Assumption 5(i) holds or \( \zeta = 0 \)

\[
\frac{\partial x_t}{\partial p} < 0, \quad \frac{\partial x_t}{\partial \tau_i^p} < 0, \quad \frac{\partial x_t}{\partial \tau_i^f} > 0,
\]

\[
\frac{\partial \tilde{s}_t}{\partial p} < 0, \quad \frac{\partial \tilde{s}_t}{\partial \tau_i^p} < 0, \quad \frac{\partial \tilde{s}_t}{\partial \tau_i^f} > 0,
\]

\[
\frac{\partial}{\partial p} \left( \frac{K_{t+1}}{Y_t} \right) < 0, \quad \frac{\partial}{\partial \tau_i^p} \left( \frac{K_{t+1}}{Y_t} \right) < 0,
\]

\[
\frac{\partial}{\partial p} \left( \frac{G_{t+1}}{Y_t} \right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \tau_i^p} \left( \frac{G_{t+1}}{Y_t} \right) > 0.
\]

Proof. The result has already been proved in more general setting in Proposition 5 when Assumption 5(i) holds.

When government invests entire proceeds in public capital, \( \zeta = 0 \) and we have

\[
\tilde{s}_t = \frac{(1-p)\beta}{1+\beta} \frac{\tau_i^p}{\tau_i^f} (1-\tau_i)(1-\alpha)
\]

\[
x_t = x(K_t, G_t) = \frac{\beta}{1+\beta} \left[ \frac{1}{\tau_i^f} - \frac{p}{\tau_i^p} \right] \frac{(1-\tau_i)}{1-\tau_i/\tau_i^f}
\]

\[
K_{t+1} = K(K_t, G_t) = \left[ \frac{\tau_i^p p^2}{\tau_i} + \frac{\tau_i^p (1-p)^2}{\tau_i^f - \tau_i} \right] \frac{\beta (1-\tau_i) W_t}{1+\beta}
\]

\[
G_{t+1} = G(K_t, G_t) = \frac{[(1+\beta)\tau_i + \beta \xi (1-\tau_i)]}{1+\beta} W_t,
\]

from which results follow by reasoning similar to that in Proposition 5 and hence these arguments are skipped here. ■

The existing static models on tax evasion show that a permanent increase in \( p \) or \( \tau_i^p \) causes a decrease in the tax evasion, \( x \). Proposition 7 shows that in this simplified version of the model proposed in this paper the tax evasion falls immediately and then remains lower along the entire equilibrium path of the economy. This is so as \( x_t \) depends only on government policy at time \( t \) and is independent of the state of the economy as discussed
earlier following Proposition 5. Generally, the effect of change in $\tau_i$ on tax evasion is ambiguous. However, due to log-preferences the effect of change in $\tau_i$ also turns out to be unambiguous.

Under Assumption 5(ii) the effect of change in government policy $\{\tau_i, \tau_i^p\}$ or of the change in probability of being caught, $p$, can be signed only when entire government revenue is used for public investment. In fact, this is a special case of Assumption 5(i) as in this case $j_t/y_t = 0$ and hence constant. However, as shown later using numerical simulations for a more general model, Proposition 7 holds for a very wide range of parameters values even under Assumption 5(ii).

### 4.2.3 Existence and Transition to Steady State

I now prove the existence and the uniqueness of the steady state with the time-invariant government policies as described earlier in this section.

**Proposition 8.** Under Assumption 1a, 3 and 4 there exist a unique steady state for this economy.

**Proof.** In the absence of tax evasion existence of the unique steady state follows from the results for the standard model. I consider the case of an interior solution when Assumption 2 holds.

The steady state is characterized by unchanging aggregate stock of the private and the public capital. Thus, the aggregate output and the share of the wage income in the aggregate output is also constant. Equations (30), (31) and the production function give the steady state stock of private and public capital and the steady state output as follows:

\[
\frac{K^*}{Y^*} = \left[\frac{\tau_i^p \beta}{\tau_i^p \beta - \tau_i^p (1 - p)^2} + \frac{\tau_i}{\tau_i - \tau_i} (1 - \beta) \right] \frac{\beta (1 - \tau_t) + \tau_t \zeta (1 - \alpha)}{1 + \beta (1 - \xi \zeta)}
\]

\[
\frac{G^*}{Y^*} = (1 - \zeta) \frac{(1 - \alpha)}{1 + \beta \xi (1 - \tau_t)}
\]

with $\xi = -\frac{1 - pr_t^p / \tau_i}{\tau_i - 1}$

\[
Y^* = \left[A \left(\frac{G^*}{Y^*}\right)^{\theta} \left(\frac{K^*}{Y^*}\right)^{\theta} \right]^{\frac{1}{1 - \theta}}
\]

The uniqueness of $K^*$, $G^*$, and $Y^*$ is immediate from these equations.

Further,

\[
K^* = (1 - p)k_{1,t+1} + pk_{2,t+1}.
\]

As per capita output is unchanging, the wage rate and the return on the capital are also unchanging and hence the consumers’ optimization problem is unchanging with time. This implies that $k_{1,t+1}$, and $k_{2,t+1}$, are constant in the steady state. We already know that savings rate, $\tilde{s}$ and $x$ are constant in the equilibrium. Hence (5), and (6) give $k_1^*$ and $k_2^*$ as

\[
k_1^* = s^*, \quad k_2^* = s^* - \tau_i^p x^* w^*
\]
where \( x^* \) is given by equation (29). Thus the distribution of private capital is also invariant in the steady state.

The equilibrium steady state distribution \( \lambda \) is

\[
\lambda(k_1^*) = (1 - p), \quad \lambda(k_2^*) = p, \quad \lambda(k) = 0 \quad \text{otherwise},
\]

and

\[
K^* = \lambda(k_1^*)k_1^* + \lambda(k_2^*)k_2^*.
\]

The case where tax on entire labor income is evaded can also be proved rigorously but is skipped here.

The results stated in Proposition 4 and Proposition 7 also hold for the comparative statics as they hold for the equilibrium in general.

It was shown earlier that with time-invariant government policies the current state of the economy summarized by private capital stock. A little algebra gives that (30) has the form

\[
K_{t+1} = bK_t^{\alpha + \theta} \quad \text{for some} \quad b > 0 \quad \text{and} \quad 0 < \alpha + \theta < 1.
\]

This implies that the economy exhibits monotone convergence of the capital stocks to their steady state values. Thus, dynamics of this economy is like the ordinary neo-classical economy. If the economy starts with capital and public infrastructure below the steady state level then they monotonically increase to their steady state values and so does the output. This further implies that the consumption when young and in two states when old also monotonically increase during transition as they are each a constant fraction of the output. This gives us

**Proposition 9.** Under Assumption 1a and 3 consumption when young and in the two states when old and stocks of both public and private capital converge monotonically to their steady state values.

5 Tax Evasion in an Open Economy

For characterization of the equilibrium and comparative dynamics of a more general model it is necessary to turn to numerical simulations. I undertake that task in the model of an economy that engages in trade but can not influence the world price of the goods. Also, it can neither borrow nor lend abroad and hence runs a balanced external account.

Though free trade is the optimal policy for such an economy in absence of distortions and externalities, in a later section, I use the model to show that in presence of tax evasion, the reduction in tariff on imported capital need not be Pareto improving.\(^{10}\) In fact, many developing countries have only very reluctantly liberalized their trade regime over the past decade and a half.

\(^{9}\)It may be noted that even if the economy starts with a ratio of two capitals that is different from that referred to in Proposition 5 the adjustment to this ratio occurs in the first period due to full depreciation.

\(^{10}\)For optimality of free trade see Dixit and Norman [10].
The open economy model has following additional features. An exogenously given fraction, $\gamma_t$, of the private capital is imported in period $t$. This is a reasonable approximation as the developing countries usually import ‘critical’ inputs and capital for which there is little scope for substitution from within the country. The government levies tariff at the rate $\tau_{et}$ on the imported capital.

The units of the domestic output and the imported capital are chosen such that, without tariff, one unit of imported capital can be bought with one unit of the domestic good. Thus, the relative real price of imported capital with respect to domestic output in absence of tariff is 1. With tariff, the price of capital for the agent, in terms of domestic output in period $t$ becomes $(1 + \gamma_t \tau_{et})$.\footnote{As the depreciation rate per period is close to 1, imported capital in this model is equivalent to imported capital and imported intermediate inputs.}

The modified budget constraints for the agent when old are

$$c_{o,t+1}^1 \leq \left[ r_{t+1} + (1 + \gamma_{t+1} \tau_{et+1})(1 - \delta_k) \right] k_{1,t+1} = \left[ r_{t+1} + (1 + \gamma_{t+1} \tau_{et+1})(1 - \delta_k) \right] \frac{s_t}{1 + \gamma_t \tau_{et}} \tag{33}$$

$$c_{o,t+1}^2 \leq \left[ r_{t+1} + (1 + \gamma_{t+1} \tau_{et+1})(1 - \delta_k) \right] k_{2,t+1} = \left[ r_{t+1} + (1 + \gamma_{t+1} \tau_{et+1})(1 - \delta_k) \right] \frac{s_t - \tau_{et} x_{t} w_{t}}{1 + \gamma_t \tau_{et}} \tag{34}$$

The government’s budget constraint for period $t$ now becomes

$$R_t = G_{t+1} - (1 - \delta_G)G_t + J_t = [\tau_{it} (1 - X_{t}) W_{t} + \tau_{it}^p p X_{t} W_{t}] + \tau_{et} \left[ \gamma_t \left\{ K_{t+1} - (1 - \delta_k)K_{t} \right\} \right], \tag{35}$$

where second term on the right hand side is the tariff revenue.

The procedure for solving the model is same as in the earlier case and hence is skipped.

Propositions 1 – 3 regarding tax evasion and existence and uniqueness of competitive equilibrium extend in a straightforward manner to this model with slight modification of Assumptions 4(i) and 4(ii). Further, Propositions 4 – 9 also hold for the corresponding simplified model with log-preferences and Cobb-Douglas production function. Thus, under Assumptions 1a, 2, 3 and suitably modified form of Assumption 4, in particular, the comparative dynamics effects of change of $p$, $\tau_{it}^p$, and $\tau_{it}$ on tax evasion are the same for this more general model with imported capital.

### 5.1 Calibration of the Model

To examine the welfare effects of trade reform, I calibrate the model to match the characteristics of the developing economies. It is assumed that prior to the tariff reform the government policy is time-invariant in the sense described in the previous section where Assumption 5 (i) holds. I allow for more general preferences

$$u(c) = \frac{c^{1-\sigma}}{1 - \sigma} \tag{36}$$

\footnote{This is a simplification that does not effect the results as the relative price of the two for the economy as whole does not change in our analysis as imposition of tariff does not change the price the economy pays for the purchase of imported capital.}

\footnote{In the data the imported capital and imported intermediate inputs together form a significant portion of imports of developing countries. Hence I restrict our attention to the tariff on the imported capital.}
and for partial depreciation of both types of capital in the model in accordance with empirical facts.

While calibrating the model parameter values used are those that are reasonable for the developing countries and based on estimates in the existing literature. The model is calibrated to match the data on \( r, p, R/Y \) which gives the values of \( \beta, x, \) and \( \tau^p \).

Durlauf and Johnson [11] study the convergence across national economies. They find that the share of the physical capital in the output/income varies between 0.30 and 0.40. The poor countries have a capital share of income in the output of 0.30, whereas for the countries with intermediate income it is 0.40. For the developed countries they find this to be 0.33. For Latin American economies, Elias [13] estimates a value of 0.50. Hence \( \alpha = 0.40 \) is a reasonable assumption. There are very different estimates of the elasticity of national output with respect to public capital varying from close to zero to 0.20 (see Lynde and Richmond [18] and Ai and Cassou [2]) and \( \theta = 0.1 \) is very reasonable.

The share of imported capital in total capital varies considerably across the developing countries and a value of 0.60 is assumed. This is also true for the public investment as a fraction of government revenue and it is assumed that the government invests 10\% of its revenues in public capital.

Each period in the model corresponds to 25 years. Assuming 5\% annual depreciation of the private capital yields \( \delta_k = 0.723 \). Assuming a 4\% annual depreciation of the public capital gives \( \delta_G = 0.64 \). A lower rate of depreciation of the public capital is justified by the fact that the private capital also consists of the plant and machinery which depreciate faster than physical infrastructure.

Dean, Desai and Riedel [8] find that the unweighted average tariff rates in India and Pakistan were 71\% and 64.8\% respectively in 1993. These rates do not include implicit tariffs due to quantitative restrictions on imports. Since developing countries have an escalated structure of protection where highest tariffs are levied on the imported consumption good, \( \tau_c = 0.60 \) is quite reasonable. A marginal tax rate on labor income of 30\% is also empirically reasonable. This is also the marginal rate that was in effect for income in the year 2001 – 2002 in India and Pakistan. However, as the model does not have imported consumption good which is major source of tariff revenue, I choose a higher value of 35\% to ascribe these tariff revenues to the labor income tax. This is an unhappy compromise.\(^{14}\)

It must emphasized that the qualitative aspects of the results are not sensitive to the values of parameters chosen above.

It is quite easy to see that in this model with tax evasion, the risk aversion characteristics of the individuals is very important. The model is calibrated for \( \sigma = 1.5 \) which is close to the micro estimates obtained by Ogaki, Ostry and Reinhart [20] and Ogaki and Reinhart[19].

\(^{14}\)If instead I chose higher rates of tariff on the imported capital, the welfare losses from the reduction of tariff, as quantified in a later section, would be even higher.

For purposes of studying the dynamics of the economy including comparative dynamics either choice gives same qualitative and similar quantitative results.
A tariff reform, as implemented in the next section, is a reduction in tariff accompanied by a corresponding change (increase) in the statutory and penal labor income tax rate so as to keep the ratio of the government revenue to the total output of the economy unchanged at the pre-reform level. With two instruments available, a rule that links them is needed. I work with a proportional penal tax rate rule (i.e. \( \frac{\tau_{it}^p}{\tau_{it}} = \text{constant} \)) which minimizes tax evasion. Yitzhaki [27] deals with proportional penal tax rate rule in a static model and shows that in this case an increase in statutory tax rate has no substitution effect (as defined in Allingham and Sandmo [3]) on tax evasion.

As stated earlier, values of \( \beta, x, \) and \( \tau_{it}^p \) are chosen to match the data on \( r, p, R/Y. \) The ratio of the government revenues to the GDP can be ascertained from Summers and Heston (1992)/ Penn World Tables (Mark 5.6a) which reveals considerable variation in the ratio. This ratio varies from 10% to 30% for the middle 90% of the countries and the average is lower for the developed countries than for the developing countries. The model is calibrated so that \( R/Y = 0.20. \) The real interest rates again vary quite widely across countries however, a 6% annual real interest rate is assumed which is on the lower side of the range of values for the developing countries. The probability of being caught, \( p, \) is fixed at 0.15 which implies that every year less than 1% of the returns are audited. This is corresponds to audit rate in some developing countries.

The values of parameters for the calibrated model are given in Table 1. The value of \( \tau_{it}^p \) implies the corresponding value of \( \chi \) shown in Table 1. The steady state of the calibrated model is summarized in Table 2. The annualized capital-output ratio is in middle of the range of values for the developing countries.\(^{15} \) The consumption age profile has a slightly downward slope in accordance with empirical observations. Agents evade tax on 24.99% of their income which is on the lower side of the estimates. The share of tariffs in government revenues at 13.3% is lower than the estimates in Tanzi [24] because the model does not have imported consumption goods which is heavily taxed in developing countries. Overall the calibrated model captures important features of the developing countries.

5.2 Transition and Comparative Dynamics of the Calibrated Model

For the simplified model discussed in section 4, Proposition 9 showed that capital stocks and consumption in all states converged monotonically to their new steady values. This also holds true for this more general open economy model as shown in Figure 2 where economy’s initial stocks of public and private capital are 60% of their steady state values. In this figure and all other subsequent figures initial steady state output is normalized to 1.

For the simplified model, Proposition 7 showed that an increase in the \( p, \) or \( \tau_{it}^p \) decreases tax evasion immediately and along entire future equilibrium path. This also hold for calibrated model as evident from Figure 3 and Figure 4 which show the response to a 10% increase in \( p \) and \( \tau_{it}^p \) respectively. From the point of view of...\(^{15} \) See Buffie [5] chapter 5.
Preferences
\[ \beta = .2004; \quad \sigma = 1.5 \]

Production Function
\[ \alpha = .4; \quad \theta = .1; \quad A = 100; \quad \delta_k = .723; \quad \delta_G = .64; \quad \gamma = .6 \]

Government Policy
\[ \tau_i = .35; \quad \tau^p_i = .705; \quad \tau_e = .6; \quad \chi = 2.014; \quad \zeta = .9 \]

Other
\[ p = .15 \]

Table 1: Parameter values for the calibrated model.

Tax Evasion
\[ x = .2499 \]

Consumption-Output Ratios
\[ c_y/y = .4676; \quad c_o/y = .4385; \quad c^1_o/y = .4885; \quad c^2_o/y = .1552 \]

Savings/Capital-Output Ratios
\[ s/y = .1548; \quad k/y = .1022; \quad k_1/y = .1138; \quad k_2/y = .0362 \]
Annualized \( k/y = 2.554 \)

Government
\[ R/y = .2; \quad J/y = .18; \quad G/y = .0313; \quad G/K = .306; \quad T/R = .133 \]

Other
\[ r = .06; \quad Y = 806.16 \]

Table 2: Steady state for the calibrated model.
policymakers, it is important that the tax evasion declines immediately rather than just across steady states. In addition, savings ratio, and capital-output ratios also behave as in case of the simplified model (see Figure 3 and Figure 4.) Thus, the characterization of the more general calibrated model is similar to the simplified model.

I undertook an elaborate calibration the model to match the characteristics of the developing economies for the purpose studying welfare effects of the trade reform that I do in a later section. However, the comparative dynamics of the model is very robust to parameter values.\textsuperscript{16} In fact, Proposition 7 establishes the comparative dynamics results under Assumption 5(ii) only under a very strong condition that $\zeta = 0$. Whereas Figure 3 and Figure 4 show that these results holds in a much more general model with partial depreciation, CRRA preferences, and an open trading economy even when only 20\% of government revenues are invested in public capital (i.e. $\zeta = .8$.)

The results from numerical simulations presented above and subsequently in the paper are arrived at by solving the actual non-linear model and hence do not have any linearization or approximation errors.\textsuperscript{17} Thus, the transition paths shown in Figure 2 – 4 are the actual paths of the relevant variables when they differ from their steady state values by a finite amount.

5.3 Other Specifications of Preferences

The models in this paper so far are restricted in one respect as compared to the models in the existing literature on tax evasion as former models allow only for constant relative risk aversion (CRRA). The comparative static effects

\textsuperscript{16}I did not discover any parameter combination for which this did not happen.

\textsuperscript{17}Given the precision of the numerical algorithms the errors arising from numerical estimation are too small to be of any consequence.
Figure 3: Comparative dynamics of the calibrated model for an increase in $p$.

Figure 4: Comparative dynamics of the calibrated model for an increase in $\tau^p_1$. 
of increase in \( p \), and \( \tau_i^p \) in existing literature are valid so long as there is decreasing absolute risk aversion (DARA) whereas CRRA preferences are a special case of DARA class. The results obtained with CRRA preferences, however, extend to models with constant absolute risk aversion (CARA) as I show now. As CRRA and CARA cases span a wide class of models with decreasing absolute risk aversion, the results of the paper are quite general.

I now consider the above open economy model with following utility function

\[
    u(c) = -b \exp\left(-c/b\right)
\]

\[
    b > 0,
\]

and \( c \) is consumption. Here \( 1/b \) is the coefficient of absolute risk aversion and the coefficient of relative risk aversion is \( c/b \). Thus, relative risk aversion rises with consumption. The calibration of this model and hence it steady state is similar to the earlier calibrated model. The value of \( b \) is determined by assuming that the coefficient of relative risk aversion when the consumption is lowest, \( i.e. \) is \( c_\sigma^2 \), equals the value of \( \sigma \) (\( = 1.5 \)) in earlier calibration.

The model then shows that transition dynamics is once again characterized by monotonicity of consumption and capital stocks. An increase in audit probability, \( p \), causes decline in tax evasion, savings rate, private capital to output ratio and rise in public capital to output ratio along the equilibrium path. The same holds for and increase in \( \tau_i^p \) as in the earlier model with CRRA preferences.

Though the results now extend to CARA preferences, the calibrated models are still restricted to cases where \( \sigma \geq 1 \) as in view of Assumption 4 an equilibrium may not exist with \( \sigma < 1 \). But, as mentioned earlier, this happens only under very special conditions. Whenever, there exists a unique competitive equilibrium with \( \sigma < 1 \), the transition and comparative dynamics are same as for \( \sigma \geq 1 \). In essence, the results in existing literature on effects of changes in audit rate and penal tax rate hold for a wider range of values of \( \sigma \) than covered by Assumption 4.

6 An Application - Welfare Effects of Trade Reform

A tariff reform is a reduction in the tariff from the existing levels along with a matching change in labor income tax rate so that revenue of the government as a fraction of the GDP is unchanged at the pre-reform level. I assume that the government removes the tariff completely.\(^{18}\)

To isolate the effect of tax evasion on welfare outcome of the trade reform, I consider a similar tax reform when there is no tax evasion. This model serves as benchmark model to compare and contrasts the effects in the actual model with tax evasion. The calibration of benchmark model is only slightly different. Matching data on the interest rate, \( r \), and \( R/Y \) now yields \( \beta = .2877 \) and \( \tau_i = .289 \). The steady state is identical to the actual

\(^{18}\)The qualitative results obtained are, however, independent of the choice of the new tariff level.
Figure 5: Welfare of current young and future generations before and after the trade reform.

economy except for the savings-output ratio which is lower, at .139, in absence of tax evasion.

In both cases, it is found that the current generations are worse off as a result of the reform. However, future generations gain from the reform. The path of the utility of the current young and the future generation in the actual model is shown in Figure 5. The path for the benchmark model is also qualitatively same. The current young, which correspond to period 1 in the figure, are unambiguously worse off in both cases with or without tax evasion. The reform induces the current young to substitute away from current consumption as the return to capital and the real interest rate falls and, in numerical simulations of the tariff reform, their expected consumption when old does not rise sufficiently causing their utility/welfare to fall. The current old are worse off too as the price of their undepreciated capital, \((1 + \gamma T_e)/g_{59}\), falls due to the reduction of tariff. Thus, both the generations that are currently alive lose from the reform.

6.1 Assessing Pareto Superiority of the Reform

However, if the removal of the tariffs is Pareto improving then the government can implement a redistributional scheme to make the reform acceptable to the current generations. This redistribution in an overlapping generations framework will take the form of a ‘backward’ intergenerational transfer that will compensate the current generations via a transfer from future generations and hence will be similar to a social security reform. The feasibility of such a transfer mechanism and hence the Pareto superiority of the tariff reform is examined next.

6.1.1 Feasibility of Intergenerational Transfers

Till now, the transfers were made to the generation from which the taxes were raised. Now individuals may receive government transfers both when young and when old. Since the output in the current period is fixed and the current old can not be made worse off and reduction of tariff encourages capital accumulation, current young can only be compensated by handing out transfers when old. This reduces the transfers made to the next generation when it is young. Thus, a chain of backward transfers starts off.

If the reform is Pareto improving the transfers made to the old will end in finite time or the transfers to the old will asymptotically reach a fraction of the total transfers. In the absence of tax evasion, such a ‘backward’
intergenerational transfer scheme can be implemented to achieve a Pareto superior outcome. Hence there are net gains from the tariff removal in the present value sense.

Figure 6 shows such a feasible intergenerational transfer scheme for this benchmark case. The intergenerational transfers continue for three periods and transfers to the old reach a maximum of $18.46\%$ of the total transfers in the second period. Intergenerational transfers slow down capital accumulation and output growth. The agents now have additional income (in the form of transfers) when old and hence reduce savings which depresses capital accumulation. When the transfers to old cease capital accumulation picks up pace again.

In the case of the tax evasion no such feasible intergenerational transfer mechanism exists for the calibrated model which indicates that the gains from trade reform are insufficient to compensate the existing generations.

### 6.1.2 Quantifying Welfare Gains or Losses

In absence of tax evasion the reform is Pareto improving. For this case, I quantify welfare effects of the reform in terms of additional consumption that can be obtained by the current young, when they are young, without making any other generation worse off.\(^{19}\) When there are losses from the reform I quantify the welfare losses in terms of minimum ‘foreign aid’ needed in the period in which the reform is undertaken. For this exercise, the intergenerational transfer scheme is modified so that all the gains from the reform are redistributed to the current young without making any other generation worse off.

In absence of tax evasion the consumption of current young, when they are young, can be increased by $59\%$ compared to its pre-reform level.\(^{20}\) This is $27\%$ of the current GDP. Instead, foreign aid of $32\%$ of the current GDP is needed to maintain the pre-reform utility level of all generations in presence of tax evasion. Thus, the presence of tax evasion tilts the balance against the reform. This happens in this model as the agents cannot diversify the risk of being caught when evading taxes.

With tax evasion, welfare losses arise because the part of the gains from trade are realized by the future

\(^{19}\)The welfare gains can be equally well calculated in terms of change in consumption of any particular generation, including current old.

\(^{20}\)This is calculated by first determining the increase in utility of the current young and then converting this increase to an equivalent increase in their consumption when young.
generations across two states in the second period of their life. Hence the certainty equivalent of the gains from the tariff reform is smaller than in the case without tax evasion. So, it is not possible to take away enough resources from future generations to compensate the current young. However, as agents now receive transfers when old in both states, they provide insurance against the risk of being caught and reduce the welfare loss.\(^{21}\)

The paper abstracts from the costs of collection of the labor income tax. As pointed by Easterly and Rebelo \([12]\), these cost are not small. In case of Canada, Vaillancourt \([26]\) finds that public and private costs associated with income tax system and social security payments is about 7% of the revenue collected. An increase in income tax rate would cause increase in these costs leading to a further welfare loss.

7 Conclusion

The paper presents a dynamic OLG model of tax evasion where government revenue is used to provide public capital that enhances the total factor productivity of the firms. The paper establishes existence and uniqueness of the competitive equilibrium for this model. It also shows that comparative static results regarding the effect of changes in probability of being caught, \(p\), and penal tax rate, \(r^p_{lt}\), derived in the existing literature for static models also hold along the entire equilibrium path for quite general preferences. An increase in audit rate or penal tax rate causes an immediate decrease in tax evasion rather than just across steady states. The result extends to economies that are small and open with respect to trade but where individuals cannot borrow or lend abroad. Further, a reduction in tax evasion causes reduction in savings and capital accumulation in the economy. When agents evade taxes they save more to ‘self-insurance’ against being caught evading taxes. A reduction in tax evasion reduces the extent of self-insurance needed.

The paper also presents a practical application of the model to study the question of trade reform in a small open economy. The model shows that in presence of tax evasion, removal of tariffs is not Pareto improving for a country that imports capital goods.

Thus, the paper extends the results in existing literature on tax evasion to a dynamic general equilibrium setting in a model where government revenues are productively employed. It also provides a tractable framework for studying issues for which tax evasion may be an important consideration. It may also be useful for studying welfare effects of policies as illustrated by the application in section 5.\(^{22}\) An immediate application of the model would be to study the welfare effects of a trade reform when a country also imports a consumption good that is not produced domestically.

\(^{21}\)I had actually considered another intergenerational transfer scheme in which the government of the developing country has full access to the international asset markets and compensates the current young and old in the period of the reform itself. The country pays the same rate of interest on the loan as the domestic equilibrium interest rate. In this case borrowing in the period of the reform is about 2-3% of the GDP and the welfare loss rises to above 1% of the current GDP.

\(^{22}\)Rampant tax evasion in developing countries may impose constraints on government policy, such as inability to increase penal tax rate proportionally with the statutory tax rate. These constraints may be important from the point of view of positive analysis. The paper also ignores the differences in costs of collection of tariffs and income taxes.
8 Appendix

Lemmas 1A-3A establish some preliminary results needed to prove Lemmas 1 – 3 in the main body of the paper.

**Lemma 1A.** Under Assumptions 1 - 3, and 4(i),

\[
\begin{align*}
    h_1 & \equiv 1 - \sigma \left(1 + \frac{F_{K,t+1}}{s_t F_{K,K,t+1}} \right) > 0 \\
    h_2 & \equiv 1 - \sigma \left(1 + \frac{F_{K,t+1}}{(s_t - \tau_{i,t} x_{t+1} w_t) F_{K,K,t+1}} \right) > 0 \\
    h_3 & \equiv (\sigma - 1) \left(F_{K,K,t+1} + \frac{\tau_i - p r_i^p}{p r_i^p} + \frac{\sigma F_{K,t+1}}{s_t - \tau_{i,t} w_i}ight) > 0, \\
    \text{and } h_4 & \equiv (\sigma - 1) \left(F_{K,K,t+1} + \frac{\tau_i - p r_i^p}{p r_i^p} + \frac{\sigma F_{K,t+1}}{p s_t - \tau_{i,t} x_{t+1} w_t} \right) > 0.
\end{align*}
\]

**Proof of Lemma 1A.** The first inequality is another way of writing Assumption 4(i). The second inequality also follows immediately from the first one as \(0 \leq s_t - \tau_{i,t} x_{t+1} w_t \leq s_t, \ F_{K,K,t+1} < 0\). The third one follows from the second inequality by noting that \(F_{K,K,t+1} > 0, \sigma_1 > 1\) and \(\tau_i > p r_i^p\). The last one is immediate consequence of the third one as \(0 < p < 1\). □

**Lemma 2A.** Under Assumption 1 and given the initial capital stocks \((K_t, G_t)\)

\[
\frac{\partial u' (c_{yt})}{\partial s_t} > 0 \quad \text{and} \quad \frac{\partial u' (c_{yt})}{\partial x_t} < 0.
\]

**Proof of Lemma 2A.** From (13) it is follows immediately that

\[
\frac{\partial c_{yt}}{\partial s_t} = -1 < 0 \quad \text{and} \quad \frac{\partial c_{yt}}{\partial x_t} = \tau_i w_t > 0
\]

and the proof is completed by noting that utility function has decreasing marginal utility by Assumption 1. □

**Lemma 3A.** Under Assumptions 1 - 3, and 4(i) and given the initial capital stocks \((K_t, G_t)\), the following hold:

1. \(\begin{align*}
    (a) & \frac{\partial u' (c_{ot+1}^1)}{\partial s_t} F_{K,t+1} < 0, \\
    (b) & \frac{\partial u' (c_{ot+1}^2)}{\partial s_t} F_{K,t+1} < 0,
\end{align*}\)

2. \(\begin{align*}
    (a) & \frac{\partial u' (c_{ot+1}^2)}{\partial x_t} F_{K,t+1} > 0, \\
    (b) & \frac{\partial}{\partial x_t} \left[(1 - p) u' (c_{ot+1}^1) F_{K,t+1} + pu' (c_{ot+1}^1) F_{K,t+1} \right] > 0.
\end{align*}\)

**Proof of Lemma 3A.** Straightforward algebra gives

\[
\frac{\partial u' (c_{ot+1}^1)}{\partial s_t} F_{K,t+1} = u' (c_{ot+1}^1) F_{K,K,t+1} h_1 < 0,
\]

as the term in big bracket is positive by Lemma 1A and as \(F_{K,K,t+1} < 0\) by Assumption 1 which proves 1(a).
Once again some algebra gives
\[ \frac{\partial u' (c_{ot+1}^2) F_{K,t+1}}{\partial s_t} = u' (c_{ot+1}^2) F_{KK,t+1} h_2 < 0, \]
as the term in big bracket is positive by Lemma 1A and as \( F_{KK,t+1} < 0 \) by Assumption 1 proving 1(b).

To prove 2(a) just repeat same procedure to obtain
\[ \frac{\partial u' (c_{ot+1}^2) F_{K,t+1}}{\partial x_t} = u' (c_{ot+1}^2) pt w_t h_4 > 0 \]
by Lemma 1A.

Finally, similar exercise for 2(b) yields
\[ \frac{\partial}{\partial x_t} \left[ (1 - p) u' (c_{ot+1}^1) F_{K,t+1} + pu' (c_{ot+1}^2) F_{K,t+1} \right] = pr^p w_t \left[ (1 - p) u' (c_{ot+1}^1) + pu' (c_{ot+1}^2) \right] \left[ (\sigma - 1) \left( \frac{F_{KK,t+1} + F_{KG,t+1}}{\sigma F_{KK,t+1}} \right) \frac{\tau_{it} - pt w_t}{\tau_{it}} \right] \]
\[ \geq pr^p w_t \left[ (1 - p) u' (c_{ot+1}^1) + pu' (c_{ot+1}^2) \right] h_3 > 0, \]
by Lemma 1A and the first inequality follows from the fact that \( u' (c_{ot+1}^2) \geq u' (c_{ot+1}^1) \).

Now, I prove Lemmas 1 – 3.

**Proof of Lemma 1.** From Lemma 2A first note that \( LHS(A) \) is increasing in \( s_t \) and decreasing in \( x_t \). We denote this as: \( LHS(A)(s_t^+, x_t^-) \). By 1(a), 1(b) and 2(b) in Lemma 3A we have \( RHS(A)(s_t^-, x_t^+) \). The two curves are depicted in Figure 7.

![Figure 7](image-url)

Consider the two sides of (A) as functions of \( s_t \) for a given value of \( x_t \). \( LHS(A) \) is increasing in \( s_t \) and \( RHS(B) \) is decreasing in \( s_t \). Further as \( s_t < \tau_{it}^p x_t w_t =: s_*, K_{t+1} = s_t - \tau_{it}^p x_t w_t \) is positive and \( c_{ot+1}^2 < 0 \). Hence \( RHS(A) \to \infty \) whereas \( LHS(A) \) is finite. Similarly, as \( s_t > [(1 - \tau_{it}) + x_t \tau_{it}] w_t + j_t =: s^* \), \( LHS(A) \not\to \infty \) and
$RHS(A)$ is finite. Thus, there exist a unique $s, s_* < s < s^*$ that solves (A) a given $x_t$. Further, an increase in $x_t$ shifts the upward sloping curve representing $LHS(A)$, downward and shifts the downward sloping curve representing $RHS(A)$, upward which unambiguously raises $s$ that solves (A) as shown in Figure 7.

The continuity of $S'_A(x_t^+)$ follows from the fact that both sides are continuous in $x_t$ and hence the curves representing the two sides shift smoothly as $x_t$ changes. Thus, the intersection of the two curves moves smoothly with $x_t$ and hence $s_t$ is a continuous function of $x_t$. ■

Proof of Lemma 2. From Lemma 2A we already have $LHS(B)(s_t^+, x_t^-)$. By 1(b) and 2(a) in Lemma 3A we have $RHS(B)(s_t^-, x_t^+)$. Once again the two curves are depicted in Figure 7. Again by the reasoning in Lemma 1, there exist a unique $s_t, s_* < s_t < s^*$ that solves (B) given $x_t$, and $s_t$ that solves (B) is increasing in $x_t$ and $S'_B(x_t^+)$ is continuous. ■

Proof of Lemma 3. With $x_t = 0$, from (A) and (B) we have

$$u'(c_{yt}) = \beta u'(c_{yt}^2) F_{k,t+1} \geq \frac{\beta P_{yt}^p}{\tau_{yt}} u'(c_{yt}^2) F_{k,t+1},$$

as $c_{yt}^1 = c_{yt}^2$. Thus, value of $RHS(B)$ is smaller than the $RHS(A)$. Hence for $s_t = s_t^{A0}, LHS(B) > RHS(B)$. As $LHS(B)$ is increasing in $s_t$ and $RHS(B)$ is decreasing in $s_t$ we have that $s_t^{B0} \leq s_t^{A0}$. ■
References


